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On the nonautonomous difference equation $x_{n+1} = A_n + \frac{x_{n-1}^p}{x_n^q}$

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ABSTRACT

In this paper we study the asymptotic behavior and the periodicity of the positive solutions of the nonautonomous difference equation:

$$x_{n+1} = A_n + \frac{x_{n-1}^p}{x_n^q}, \quad n = 0, 1, \ldots,$$

where A_n is a positive bounded sequence, $p, q \in (0, \infty)$ and x_{-1}, x_0 are positive numbers.

1. Introduction

In the papers [1–6,9,11,22,24,25,28,30,31] the authors investigated the equation

$$x_n = A + \frac{x_{n-k}^p}{x_{n-m}^q}, \quad n = 0, 1, \ldots, \tag{1.1}$$

where $A, p, q \in [0, \infty)$ and $k, m \in N$, $k \neq m$. Moreover there exist many other papers related with Eq. (1.1) and on its extensions (see [13,14,26,27,29,32–35]).

In addition in the papers [10,16,23] the authors studied the behavior of the positive solutions of the difference equation

$$x_{n+1} = p_n + \frac{x_{n-1}}{x_n}, \quad n = 0, 1, \ldots, \tag{1.2}$$

where p_n is a bounded positive sequence and the initial values x_{-1}, x_0 are positive numbers. We note that the papers [18–20] are devoted to Eq. (1.2) and on its some extensions.

Finally in [21] the authors investigated the behavior of the positive solutions of the difference equation

$$x_{n+1} = A_n + \left(\frac{x_{n-1}}{x_n}\right)^p, \quad n = 0, 1, \ldots,$$

where A_n is a bounded positive sequence, $p \in (0, 1) \cup (1, \infty)$ and the initial values x_{-1}, x_0 are positive numbers.

Motivated by the above papers we study the attractivity, the periodicity and the stability of the positive solutions of the difference equation

$$x_{n+1} = A_n + \frac{x_{n-1}^p}{x_n^q}, \quad n = 0, 1, \ldots, \tag{1.3}$$

where A_n is bounded positive sequence, p, q are positive constants and x_{-1}, x_0 are positive numbers.

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We note that since difference equations have many applications in applied sciences there are a lot of papers concerning difference equations and their applications (see for example [1–31]). Finally some recent applications of linear as well as nonlinear difference equations are included in the papers [7,8].

2. Asymptotic behavior of the positive solutions

In this section we find conditions so that if $\bar{x}_n$ is a fixed solution of (1.3) then $\bar{x}_n$ attracts all solutions of (1.3). Let

$$y_n = \frac{x_n}{\bar{x}_n}, \quad n = -1, 0, 1, \ldots. \tag{2.1}$$

Then from (1.3) and (2.1) we take that y_n satisfies the difference equation

$$y_{n+1} = \frac{A_n + \frac{\bar{x}^p_{n-1}}{\bar{x}^q_n}\frac{y^p_{n-1}}{y^q_n}}{A_n + \frac{\bar{x}^p_{n-1}}{\bar{x}^q_n}}. \tag{2.2}$$

To prove the first result of this paper we need the following lemmas.

Lemma 2.1. *Let y_n be a particular positive solution of (2.2) . Suppose that there exists an $m \in \{0, 1, \ldots\}$ such that*

$$y_{2m-1} \geqslant 1, \quad y_{2m} < 1. \tag{2.3}$$

Then

$$y^p_{2n-1} > 1, \quad y^q_{2n} < 1, \quad y^q_{2n-1} > 1, \quad y^p_{2n} < 1, \quad n = m+1, m+2, \ldots. \tag{2.4}$$

In addition if

$$y_{2m-1} < 1, \quad y_{2m} \geqslant 1, \tag{2.5}$$

then

$$y^p_{2n-1} < 1, \quad y^q_{2n} > 1, \quad y^q_{2n-1} < 1, \quad y^p_{2n} > 1, \quad n = m+1, m+2, \ldots. \tag{2.6}$$

The proof of Lemma 2.1 follows immediately from (2.2).

Lemma 2.2. *Consider the function*

$$F(x, y, z) = \frac{z + xy}{z + x}, \quad x, y, z > 0. \tag{2.7}$$

Then the following statements are true:

(i) *F is an increasing function in y for any x, $z \in (0, \infty)$;*
(ii) *F is an increasing (resp. decreasing) function in x for any $y \in (1, \infty)$ (resp. $y \in (0, 1)$) and $z \in (0, \infty)$;*
(iii) *F is an increasing (resp. decreasing) function in z for any $y \in (0, 1)$ (resp. $y \in (1, \infty)$) and $x \in (0, \infty)$.*

Proof. Statement (i) is obvious. Since

$$\frac{\partial F}{\partial x} = \frac{z(y-1)}{(x+z)^2}, \quad \frac{\partial F}{\partial z} = \frac{x(1-y)}{(x+z)^2},$$

the proof of statements (ii) and (iii) follows immediately. □

Using the same argument to prove Proposition 2.1 of [21] and using Theorem 2.6.2 of [15] we take the following lemma.

Lemma 2.3. *Suppose that A_n is a bounded sequence such that*

$$0 < m = \liminf_{n\to\infty} A_n, \quad M = \limsup_{n\to\infty} A_n < \infty. \tag{2.8}$$

Suppose also that

$$0 < p < 1. \tag{2.9}$$

Then every positive solution of Eq. (1.3) is bounded and persists.

We state now our proposition.

Proposition 2.1. *Consider Eq. (1.3) where A_n is a bounded positive sequence such that (2.8) holds. Suppose also that*

$$0 < p + q < 1, \quad q > p. \tag{2.10}$$

Let $\bar{x}_n$ be a fixed solution of (1.3) and x_n be an arbitrary solution of (1.3). Then

$$\lim_{n\to\infty} y_n = 1, \tag{2.11}$$

where y_n is defined in (2.1).

Proof. Using Lemma 2.3 and relation (2.1) we have

$$\begin{aligned} &0 < \eta = \liminf_{n\to\infty} y_n, \quad \theta = \limsup_{n\to\infty} y_n < \infty, \\ &0 < k_1 = \liminf_{n\to\infty} \bar{x}_n, \quad k_2 = \limsup_{n\to\infty} \bar{x}_n < \infty. \end{aligned} \tag{2.12}$$

We suppose that there exists an $m \in \{1,2,\ldots\}$ such that either (2.3) or (2.5) holds. Consider that (2.3) is satisfied. Using (2.2) we obtain that for $n \geqslant m$

$$y_{2n+1} = \frac{A_{2n} + \frac{\bar{x}_{2n-1}^p}{\bar{x}_{2n}^q}\frac{y_{2n-1}^p}{y_{2n}^q}}{A_{2n} + \frac{\bar{x}_{2n-1}^p}{\bar{x}_{2n}^q}}, \quad y_{2n+2} = \frac{A_{2n+1} + \frac{\bar{x}_{2n}^p}{\bar{x}_{2n+1}^q}\frac{y_{2n}^p}{y_{2n+1}^q}}{A_{2n+1} + \frac{\bar{x}_{2n}^p}{\bar{x}_{2n+1}^q}}. \tag{2.13}$$

Then from Lemmas 2.1 and 2.2, and relations (2.8), (2.12) and (2.13) we have

$$\theta\eta^q \leqslant \frac{m\eta^q + k\theta^p}{m+k}, \quad \theta^q\eta \geqslant \frac{m\theta^q + k\eta^p}{m+k}, \quad k = \frac{k_2^p}{k_1^q},$$

and so

$$m\eta^q\theta^{q-1} + k\theta^{p+q-1} \geqslant m\theta^q\eta^{q-1} + k\eta^{p+q-1}. \tag{2.14}$$

Relations (2.10) and (2.14) imply that

$$m\eta^{q-1}\theta^{q-1}(\theta-\eta) \leqslant k(\theta^{p+q-1} - \eta^{p+q-1}) = k(\eta\theta)^{-1+p+q}(\eta^{1-p-q} - \theta^{1-p-q}) \leqslant 0,$$

and so we have that $\theta = \eta$ which implies that there exists $\lim_{n\to\infty} y_n$. Using (2.4) we have that (2.11) is true. Similarly we can prove that if (2.5) is satisfied then (2.11) holds. Suppose now that neither (2.3) nor (2.5) holds. Then from Lemma 2.1 we get

$$y_n < 1, \quad \text{or} \quad y_n \geqslant 1, \qquad n \geqslant -1. \tag{2.15}$$

Without loss of generality we may suppose that

$$y_n < 1, \quad n \geqslant -1. \tag{2.16}$$

We claim that

$$y_{n+1}^q > y_n^p, \quad n \geqslant -1. \tag{2.17}$$

Suppose on the contrary that there exists a $v \geqslant -1$ such that

$$y_{v+1}^q \leqslant y_v^p. \tag{2.18}$$

Then from (2.2) and (2.18) we get

$$y_{v+2} = \frac{A_{v+1} + \frac{\bar{x}_v^p}{\bar{x}_{v+1}^q}\frac{y_v^p}{y_{v+1}^q}}{A_{v+1} + \frac{\bar{x}_v^p}{\bar{x}_{v+1}^q}} \geqslant 1,$$

which contradicts to (2.16). So (2.17) is true. Using relations (2.10), (2.16) and (2.17) we get

$$y_{n+1}^q > y_n^p > y_n^q, \quad n \geqslant -1,$$

which implies that

$$y_{n+1} > y_n, \quad n \geqslant -1. \tag{2.19}$$

Moreover from (2.2) we have for $n \geqslant 0$

$$|y_{n+1} - 1| = \frac{\frac{\bar{x}_{n-1}^p}{\bar{x}_n^q}}{A_n + \frac{\bar{x}_{n-1}^p}{\bar{x}_n^q}}\left|\frac{y_{n-1}^p}{y_n^q} - 1\right| < \left|\frac{y_{n-1}^p}{y_n^q} - 1\right|. \tag{2.20}$$

Using (2.16), (2.17) and (2.20) we get

$$1 - y_{n+1} < 1 - \frac{y_{n-1}^p}{y_n^q}, \quad n \geqslant 0. \tag{2.21}$$

In addition from (2.16) and (2.19) we have that

$$\lim_{n\to\infty} y_n = \lambda \leqslant 1. \tag{2.22}$$

Hence relations (2.21) and (2.22) imply that

$$\lambda^{q-p+1} \geqslant 1,$$

and so from (2.10) and (2.22) we have that $\lambda = 1$. Similarly if the second relation of (2.15) is satisfied we can prove that $\lambda = 1$. This completes the proof of the proposition. □

3. Periodicity and stability

In the next proposition we find sufficient conditions for the existence, the uniqueness of 2-periodic and 3-periodic solutions for Eq. (1.3) and the convergence of the positive solutions of (1.3) to the periodic solutions.

Proposition 3.1. *Consider Eq. (1.3). Then the following statements are true:*

(i) *Suppose that A_n is a positive two-periodic sequence such that*

$$A_{n+2} = A_n, \quad n = 0, 1, \ldots. \tag{3.1}$$

Suppose also that (2.10) are satisfied. Then Eq. (1.3) has a unique two periodic solution and every positive solution of (1.3) tends to the unique 2-periodic solution.

(ii) *Suppose that A_n is a positive periodic sequence of period three such that*

$$A_{n+3} = A_n, \quad n = 0, 1, \ldots. \tag{3.2}$$

Suppose also that p, q satisfy (2.10) and there exists a positive number ϵ and a $\theta \in (0, \frac{1}{2})$ such that

$$\frac{(B+\epsilon)^p}{C^q} < \epsilon, \quad \frac{pq}{C^{2(q+1-p)}} + \frac{q\epsilon}{C} < \theta, \quad \frac{p}{C^{q+1-p}} + \frac{q^2\epsilon}{C^{q+2-p}} < \theta, \tag{3.3}$$

where

$$B = \max\{A_0, A_1, A_2\}, \quad C = \min\{A_0, A_1, A_2\}. \tag{3.4}$$

Then Eq. (1.3) has a unique periodic solution of period three and every positive solution of (1.3) tends to the unique 3-periodic solution.

Proof. (i) First we prove that (1.3) has a unique 2-periodic solution. Let x_n be a solution of (1.3). Using (3.1), x_n is periodic of period two if and only if the initial values x_{-1}, x_0 satisfy

$$x_{-1} = x_1 = A_0 + \frac{x_{-1}^p}{x_0^q}, \quad x_0 = x_2 = A_1 + \frac{x_0^p}{x_1^q}. \tag{3.5}$$

We set $x_{-1} = x$, $x_0 = y$ then from (3.5) we obtain the system of equations

$$x = A_0 + \frac{x^p}{y^q}, \quad y = A_1 + \frac{y^p}{x^q}. \tag{3.6}$$

We prove that (3.6) has a solution $(\bar{x}, \bar{y})$, $\bar{x} > 0$, $\bar{y} > 0$. From the first relation of (3.6) we get

$$y = \frac{x^{\frac{p}{q}}}{(x - A_0)^{\frac{1}{q}}}. \tag{3.7}$$

From (3.7) and the second relation of (3.6) we have

$$\frac{x^{\frac{p}{q}}}{(x - A_0)^{\frac{1}{q}}} - A_1 - \frac{x^{\frac{p^2-q^2}{q}}}{(x - A_0)^{\frac{p}{q}}} = 0. \tag{3.8}$$

We consider the function

$$f(x) = \frac{x^{\frac{p}{q}}}{(x - A_0)^{\frac{1}{q}}} - A_1 - \frac{x^{\frac{p^2-q^2}{q}}}{(x - A_0)^{\frac{p}{q}}}. \tag{3.9}$$

From (3.9) we get

$$f(x) = \frac{1}{(x-A_0)^{\frac{p}{q}}}\left(\frac{x^{\frac{p}{q}}}{(x-A_0)^{\frac{1-p}{q}}} - x^{\frac{p^2-q^2}{q}}\right) - A_1. \tag{3.10}$$

From relations (2.10), (3.9) and (3.10) we obtain

$$\lim_{x\to A_0+} f(x) = \infty, \quad \lim_{x\to\infty} f(x) = -A_1. \tag{3.11}$$

So Eq. (3.9) has a solution $\bar{x} > A_0$. Then if

$$\bar{y} = \frac{\bar{x}^{\frac{p}{q}}}{(\bar{x}-A_0)^{\frac{1}{q}}},$$

we have that the solution $\bar{x}_n$ of (1.3) with initial values $x_{-1} = \bar{x},\ x_0 = \bar{y}$ is a periodic solution of period two. Finally using Proposition 2.1 it is obvious that x_n is the unique periodic solution of period two and every positive solution of (1.3) tends to the unique periodic solution of period two.

(ii) Using (3.2) x_n is a 3-periodic solution of (1.3) if the initial values x_{-1}, x_0 satisfy

$$x_2 = x_{-1} = A_1 + \frac{x_0^p}{x_1^q}, \quad x_3 = x_0 = A_2 + \frac{x_1^p}{x_{-1}^q}. \tag{3.12}$$

We set $x_{-1} = x$, $x_0 = y$ in (3.12) and we take the system of nonlinear equations

$$x = A_1 + \frac{y^p}{(h(x,y))^q}, \quad y = A_2 + \frac{(h(x,y))^p}{x^q}, \quad h(x,y) = A_0 + \frac{x^p}{y^q}. \tag{3.13}$$

We consider the function

$$H : [A_1, A_1+\epsilon] \times [A_2, A_2+\epsilon] \to IR,$$

such that

$$H(x,y) = (f(x,y), g(x,y)), \quad f(x,y) = A_1 + \frac{y^p}{(h(x,y))^q}, \quad g(x,y) = A_2 + \frac{(h(x,y))^p}{x^q}. \tag{3.14}$$

First we prove that H is in $[A_1,A_1+\epsilon] \times [A_2,A_2+\epsilon]$. Obviously

$$A_1 < f(x,y), \quad A_2 < g(x,y), \quad (x,y) \in [A_1, A_1+\epsilon] \times [A_2, A_2+\epsilon]. \tag{3.15}$$

Moreover from (3.3), (3.4) and (3.13) we get for $(x,y) \in [A_1,A_1+\epsilon] \times [A_2,A_2+\epsilon]$

$$f(x,y) = A_1 + \frac{y^p}{(h(x,y))^q} \leqslant A_1 + \frac{(A_2+\epsilon)^p}{\left(A_0 + \frac{x^p}{y^q}\right)^q} \leqslant A_1 + \frac{(A_2+\epsilon)^p}{A_0^q} \leqslant A_1 + \frac{(B+\epsilon)^p}{C^q} < A_1 + \epsilon, \tag{3.16}$$

$$g(x,y) = A_2 + \frac{(h(x,y))^p}{x^q} \leqslant A_2 + \frac{\left(B + \frac{(B+\epsilon)^p}{C^q}\right)^p}{C^q} < A_2 + \frac{(B+\epsilon)^p}{C^q} < A_2 + \epsilon. \tag{3.17}$$

Therefore from (3.14)–(3.17) we have that H is in $[A_1,A_1+\epsilon] \times [A_2,A_2+\epsilon]$. We prove that H is a contraction in $[A_1,A_1+\epsilon] \times [A_2,A_2+\epsilon]$. We prove that

$$\left|\frac{\partial f}{\partial x}\right| < \theta, \quad \left|\frac{\partial f}{\partial y}\right| < \theta, \quad \left|\frac{\partial g}{\partial x}\right| < \theta, \quad \left|\frac{\partial g}{\partial y}\right| < \theta. \tag{3.18}$$

From (3.14) we get

$$\begin{aligned} &\frac{\partial f}{\partial x} = -\frac{pq}{y^{q-p}x^{1-p}(h(x,y))^{q+1}}, \quad \frac{\partial f}{\partial y} = \frac{p}{y^{1-p}(h(x,y))^q} + \frac{q^2x^p}{y^{q+1-p}(h(x,y))^{q+1}}, \\ &\frac{\partial g}{\partial x} = -\frac{q(h(x,y))^p}{x^{q+1}} + \frac{p^2}{y^q x^{q+1-p}(h(x,y))^{1-p}}, \quad \frac{\partial g}{\partial y} = -\frac{pq}{x^{q-p}y^{1+q}(h(x,y))^{1-p}}. \end{aligned} \tag{3.19}$$

From (2.10), (3.3), (3.4), (3.13) and (3.19) we get for all $(x,y) \in [A_1, A_1+\epsilon] \times [A_2, A_2+\epsilon]$

$$\begin{aligned}
&\left|\frac{\partial f}{\partial x}\right| < \frac{pq}{C^{q-2p+1}A_0^{q+1}} < \frac{pq}{C^{2(q-p+1)}} < \theta,\\
&\left|\frac{\partial f}{\partial y}\right| < \frac{p}{C^{1-p}A_0^{q}} + \frac{q^2(B+\epsilon)^p}{C^{q-p+1}A_0^{q+1}} < \frac{p}{C^{q-p+1}} + \frac{\epsilon q^2}{C^{q+2-p}} < \theta,\\
&\left|\frac{\partial g}{\partial x}\right| < \frac{q\left(A_0+\frac{(A_1+\epsilon)^p}{A_2^q}\right)^p}{C^{q+1}} + \frac{p^2}{C^{2q-p+1}A_0^{1-p}} < \frac{q\left(B+\frac{(B+\epsilon)^p}{C^q}\right)^p}{C^{q+1}} + \frac{p^2}{C^{2(q-p+1)}} < \frac{q(B+\epsilon)^p}{C^{q+1}} + \frac{p^2}{C^{2(q-p+1)}} < \frac{\epsilon q}{C} + \frac{pq}{C^{2(q-p+1)}} < \theta,\\
&\left|\frac{\partial g}{\partial y}\right| < \frac{pq}{C^{2q-p+1}A_0^{1-p}} < \frac{pq}{C^{2(q-p+1)}} < \theta.
\end{aligned} \tag{3.20}$$

Therefore from (3.20) relations (3.18) are true. Moreover there exist $\xi_i \in [A_1, A_1+\epsilon]$, $\eta_i \in [A_2, A_2+\epsilon]$, $i = 1, 2$ such that for all $x_1, x_2 \in [A_1, A_1+\epsilon]$ and $y_1, y_2 \in [A_2, A_2+\epsilon]$

$$\begin{aligned}
f(x_1,y_1) - f(x_1,y_2) &= \frac{\partial f(x_1,\eta_1)}{\partial y}(y_1-y_2),\\
f(x_1,y_2) - f(x_2,y_2) &= \frac{\partial f(\xi_1,y_2)}{\partial x}(x_1-x_2),\\
g(x_1,y_1) - g(x_1,y_2) &= \frac{\partial g(x_1,\eta_2)}{\partial y}(y_1-y_2),\\
g(x_1,y_2) - g(x_2,y_2) &= \frac{\partial g(\xi_2,y_2)}{\partial x}(x_1-x_2).
\end{aligned} \tag{3.21}$$

Relations (3.18) and (3.21) imply that

$$|f(x_1,y_1) - f(x_2,y_2)| \leqslant |f(x_1,y_1) - f(x_1,y_2)| + |f(x_1,y_2) - f(x_2,y_2)| \leqslant 2\theta \max\{|x_1-x_2|, |y_1-y_2|\},$$

$$|g(x_1,y_1) - g(x_2,y_2)| \leqslant |g(x_1,y_1) - g(x_1,y_2)| + |g(x_1,y_2) - g(x_2,y_2)| \leqslant 2\theta \max\{|x_1-x_2|, |y_1-y_2|\},$$

and so

$$\max\{|f(x_1,y_1) - f(x_2,y_2)|, |g(x_1,y_1) - g(x_2,y_2)|\} \leqslant 2\theta \max\{|x_1-x_2|, |y_1-y_2|\}. \tag{3.22}$$

So from (3.22) and since $\theta \in \left(0, \frac{1}{2}\right)$ the function H is a contraction in $[A_1, A_1+\epsilon] \times [A_2, A_2+\epsilon]$. Hence by Theorem 1.7.1 (Banach Contraction Principle) (see [15]) there exists a unique $(\bar{x}, \bar{y}) \in [A_1, A_1+\epsilon] \times [A_2, A_2+\epsilon]$ such that

$$\bar{x} = f(\bar{x}, \bar{y}), \quad \bar{y} = g(\bar{x}, \bar{y}).$$

Therefore the solution x_n with initial values $x_{-1} = \bar{x},\ x_0 = \bar{y}$ is a periodic solution of period three. Using Proposition 2.1 it is obvious that x_n is the unique solution of period three and every positive solution of (1.3) tends to the unique 3-periodic solution of (1.3) as $n \to \infty$. This completes the proof of the proposition. □

In the last proposition we study the stability of the unique periodic solution of Eq. (1.3).

Proposition 3.2. *Consider Eq. (1.3) . Then the following statements are true:*

(i) *Suppose that relations (2.10) and (3.1) are satisfied. Suppose also that*

$$\frac{p}{A_1^q A_0^{1-p}} + \frac{p^2+q^2}{(A_1A_0)^{q+1-p}} + \frac{p}{A_0^q A_1^{1-p}} < 1. \tag{3.23}$$

Then the unique 2-periodic solution of (1.3) is globally asymptotically stable.

(ii) *Suppose that (2.10), (3.2) and (3.3) hold. Suppose also that*

$$\frac{3pq}{C^{2(p+q-1)}} + \frac{p^3+q^3}{C^{3(p+q-1)}} < 1. \tag{3.24}$$

Then the unique 3-periodic solution of (1.3) is globally asymptotically stable.

Proof. (i) From Proposition 3.1 there exists a unique periodic solution $\bar{x}_n$ of Eq. (1.3) of period two. Let

$$x_{2n-1} = \bar{x}, \quad x_{2n} = \bar{y}, \quad n = 0, 1, \ldots.$$

From (1.3) we get

$$x_{2n+1} = A_0 + \frac{x_{2n-1}^p}{x_{2n}^q}, \quad x_{2n+2} = A_1 + \frac{x_{2n}^p}{x_{2n+1}^q}. \tag{3.25}$$

If we set $x_{2n-1} = z_n$, $x_{2n} = w_n$ in (3.25) we get

$$z_{n+1} = A_0 + \frac{z_n^p}{w_n^q}, \quad w_{n+1} = A_1 + \frac{w_n^p}{z_{n+1}^q}. \tag{3.26}$$

Then $(\bar{x}, \bar{y})$ is the positive equilibrium of (3.26) and the linearized system of (3.26) about $(\bar{x}, \bar{y})$ is the system:

$$v_{n+1} = Bv_n,$$

where:

$$B = \begin{pmatrix} \frac{p}{\bar{x}^{1-p}\bar{y}^q} & \frac{-q\bar{x}^p}{\bar{y}^{q+1}} \\ \frac{-pq}{\bar{x}^{q+2-p}\bar{y}^{q-p}} & \frac{p}{\bar{x}^q\bar{y}^{1-p}} + \frac{q^2}{(\bar{x}\bar{y})^{q+1-p}} \end{pmatrix}, \quad v_n = \begin{pmatrix} z_n \\ w_n \end{pmatrix}.$$

The characteristic equation of B is the following

$$\lambda^2 - \lambda\left(\frac{p}{\bar{x}^{1-p}\bar{y}^q} + \frac{p}{\bar{x}^q\bar{y}^{1-p}} + \frac{q^2}{(\bar{x}\bar{y})^{q+1-p}}\right) + \frac{p^2}{(\bar{x}\bar{y})^{q+1-p}} = 0. \tag{3.27}$$

Since $\bar{x}, \ \bar{y}$ satisfy (3.6) we have $\bar{x} > A_0, \ \bar{y} > A_1$ and so relation (3.23) implies that

$$\frac{p}{\bar{x}^{1-p}\bar{y}^q} + \frac{p}{\bar{x}^q\bar{y}^{1-p}} + \frac{p^2+q^2}{(\bar{x}\bar{y})^{q+1-p}} < \frac{p}{A_1^q A_0^{1-p}} + \frac{p^2+q^2}{(A_1A_0)^{q+1-p}} + \frac{p}{A_0^q A_1^{1-p}} < 1.$$

So from Remark 1.3.1 of [15] all the roots of (3.27) are of modulus less than 1. Hence $(\bar{x}, \bar{y})$ is locally asymptotically stable. Finally using Proposition 3.1 we have that the unique 2-periodic solution of (1.3) is globally asymptotically stable. This completes the proof of the statement (i).

(ii) From Proposition 2.1 there exists a unique 3-periodic solution $\bar{x}_n$ of (1.3). Let

$$x_{3n-1} = \bar{x}, \quad x_{3n} = \bar{y}, \quad x_{3n+1} = A_0 + \frac{\bar{x}^p}{\bar{y}^q} = \bar{z}, \quad n = 0, 1, \ldots.$$

From (1.3) we get

$$x_{3n+1} = A_0 + \frac{x_{3n-1}^p}{x_{3n}^q}, \quad x_{3n+2} = A_1 + \frac{x_{3n}^p}{x_{3n+1}^q}, \quad x_{3n+3} = A_2 + \frac{x_{3n+1}^p}{x_{3n+2}^q}, \quad n = 0, 1, \ldots. \tag{3.28}$$

If we set $x_{3n-2} = u_n$, $x_{3n-1} = v_n$, $x_{3n} = w_n$ in (3.28) we get

$$u_{n+1} = A_0 + \frac{v_n^p}{w_n^q}, \quad v_{n+1} = A_1 + \frac{w_n^p}{u_{n+1}^q}, \quad w_{n+1} = A_2 + \frac{u_{n+1}^p}{v_{n+1}^q}, \qquad n = 0, 1, \ldots. \tag{3.29}$$

Then $(\bar{z}, \bar{x}, \bar{y})$ is the positive equilibrium of (3.29) and the linearized system of (3.29) about the equilibrium $(\bar{z}, \bar{x}, \bar{y})$ is the following

$$z_{n+1} = Tz_n, \quad T = \begin{pmatrix} 0 & r_1 & s_1 \\ 0 & r_2 & s_2 \\ 0 & r_3 & s_3 \end{pmatrix}, \quad z_n = \begin{pmatrix} u_n \\ v_n \\ w_n \end{pmatrix},$$

where

$$r_1 = \frac{p}{\bar{x}^{1-p}\bar{y}^q}, \quad s_1 = -\frac{q\bar{x}^p}{\bar{y}^{q+1}}, \quad r_2 = -\frac{pq}{\bar{x}^{1-p}\bar{y}^{q-p}\bar{z}^{q+1}},$$
$$s_2 = \frac{p}{\bar{y}^{1-p}\bar{z}^q} + \frac{q^2\bar{x}^p}{\bar{y}^{q+1-p}\bar{z}^{q+1}}, \quad r_3 = \frac{p^2}{\bar{x}^{q+1-p}\bar{y}^q\bar{z}^{1-p}} + \frac{pq^2}{\bar{x}^{2+q-p}\bar{y}^{q-p}\bar{z}^{q+1-p}},$$
$$s_3 = -\frac{pq}{\bar{x}^{q-p}\bar{y}^{q+1}\bar{z}^{1-p}} - \frac{pq}{\bar{x}^{1+q}\bar{y}^{1-p}\bar{z}^{q-p}} - \frac{q^3}{(\bar{x}\bar{y}\bar{z})^{q+1-p}}.$$

The characteristic equation of the matrix T is

$$\lambda\left(\lambda^2 + \lambda\left(\frac{pq}{\bar{x}^{q-p}\bar{y}^{q+1}\bar{z}^{1-p}} + \frac{pq}{\bar{x}^{1-p}\bar{y}^{q-p}\bar{z}^{q+1}} + \frac{pq}{\bar{x}^{1+q}\bar{y}^{1-p}\bar{z}^{q-p}} + \frac{q^3}{(\bar{x}\bar{y}\bar{z})^{q+1-p}}\right) - \frac{p^3}{(\bar{x}\bar{y}\bar{z})^{q+1-p}}\right) = 0. \tag{3.30}$$

Using (2.10) and (3.24) and Remark 1.3.1 of [15] all the roots of (3.30) are of modulus less than 1. Hence the unique 3-periodic solution of (1.3) is locally asymptotically stable. Finally from Proposition 3.1 the unique 3-periodic solution is globally asymptotically stable. This completes the proof of the proposition. □

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