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On the system of two difference equations of exponential form: $x_{n+1} = a + bx_{n-1}e^{-y_n}, y_{n+1} = c + dy_{n-1}e^{-x_n}$

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1. Introduction

In [1], the authors studied the boundedness, the asymptotic behavior, the periodic character of the solutions and the stability character of the positive equilibrium of the difference equation

 $x_{n+1} = a + b x_{n-1} \mathrm{e}^{-x_n},$

where a, b are positive constants and the initial values x_{-1} , x_0 are positive numbers. Furthermore, in [1] the authors used a as the immigration rate and b as the growth rate in the population model. In fact, this was a model suggested by the people from the Harvard School of Public Health; studying the population dynamics of one species x_n .

Motivated by the above paper we will extend the above difference equation to a system of difference equations; our goal will be to investigate the boundedness, the persistence and the asymptotic behavior of the positive solutions of the system of two difference equations of exponential form

$$x_{n+1} = a + bx_{n-1}e^{-y_n},$$

$$y_{n+1} = c + dy_{n-1}e^{-x_n}$$
(1.1)

where *a*, *b*, *c*, *d* are positive constants and the initial values x_{-1} , x_0 , y_{-1} , y_0 are positive real values; also, *b* will be growth rate of species x_n and *d* will be the growth rate of species y_n ; in addition, *a* will be the immigration rate into species x_n and *c* will be the immigration rate into species y_n . Furthermore, system (1.1) can be applied as a two directional interactive and invasive species model where species x_n and y_n affect each other's population in both directions. Observe that it is very crucial for every positive solution of system (1.1) to be bounded as the population of species x_n and y_n cannot grow infinitely large due to the limited resources. Moreover, the equilibrium point (\bar{x}, \bar{y}) of system (1.1) is considered to be the natural ideal

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ABSTRACT

It is the goal of this paper to study the boundedness, the persistence and the asymptotic behavior of the positive solutions of the system of two difference equations of exponential form

$$x_{n+1} = a + bx_{n-1}e^{-y_n}, \qquad y_{n+1} = c + dy_{n-1}e^{-x_n}$$

where a, b, c, d are positive constants, and the initial values x_{-1}, x_0, y_{-1}, y_0 are positive real values.

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population; that is, if the population is left alone then it remains there. Therefore, convergence to the equilibrium point (\bar{x}, \bar{y}) will apply that the population of both species tends to the natural ideal population.

Difference equations and systems of difference equations of exponential form can be found in the following papers: [2,1,3–8]. Moreover, as difference equations have many applications in applied sciences, there are many papers and books that can be found concerning the theory and applications of difference equations; (for partial review of the theory of difference equations, systems of difference equations and their applications see [9–27,2,28,1,29,4,30–35,7,36,8,37–39] and the references cited therein).

2. Asymptotic behavior of the solutions of system (1.1)

First it is very crucial to establish the boundedness and persistence of solutions; in the first proposition we will study the boundedness and persistence of the positive solutions of system (1.1) by comparing them with solutions of a solvable system of difference equations. Our method is a modification of the method in Theorem 2 in [40]. For related and similar results see, [9,15,16,24,25,2,30–35,7,36].

Proposition 2.1. Consider system (1.1) such that:

$$be^{-c} < 1, \quad de^{-a} < 1.$$
 (2.1)

Then every positive solution of (1.1) is bounded and persists.

Proof. Let (x_n, y_n) be an arbitrary solution of (1.1). Thus from (1.1) we see that

 $x_n \ge a, \quad y_n \ge c, \quad n = 1, 2, \dots$ (2.2)

In addition, it follows from (1.1) and (2.2) that

$$x_{n+1} \le a + bx_{n-1}e^{-c}, \qquad y_{n+1} \le c + dy_{n-1}e^{-a}, \quad n = 0, 1, \dots$$
 (2.3)

We will now consider the non-homogeneous difference equations

$$z_{n+1} = a + bz_{n-1}e^{-c}, \quad v_{n+1} = c + dv_{n-1}e^{-a}, \quad n = 0, 1, \dots$$
 (2.4)

Therefore, from (2.4) an arbitrary solution (z_n, v_n) of (2.4) is given by

$$z_n = r_1 (be^{-c})^{n/2} + r_2 (-1)^n (be^{-c})^{n/2} + \frac{a}{1 - be^{-c}}, \quad n = 1, 2, \dots$$

$$v_n = s_1 (de^{-a})^{n/2} + s_2 (-1)^n (de^{-c})^{n/2} + \frac{c}{1 - de^{-a}}, \quad n = 1, 2, \dots$$
(2.5)

where r_1 , r_2 , s_1 , s_2 depend on the initial values z_{-1} , z_0 , v_{-1} , v_0 . Thus we see that relations (2.1) and (2.5) imply that z_n and w_n are bounded sequences. Now we will consider the solution (z_n , v_n) of (2.4) such that

$$z_{-1} = x_{-1}, \quad z_0 = x_0, \quad v_{-1} = y_{-1}, \quad v_0 = y_0.$$
 (2.6)

Thus from (2.3) and (2.6) we get

$$x_n \le z_n, \quad y_n \le v_n, \quad n = 1, 2, \dots$$
 (2.7)

Therefore it follows that x_n , y_n are bounded sequences. Hence from (2.2) the proof of the proposition is now complete.

In the next proposition we will study the existence of invariant intervals of system (1.1).

Proposition 2.2. Consider system (1.1) where relations (2.1) hold. Then the following statements are true:

(i) The set

$$\left[a, \frac{a}{1-b\mathrm{e}^{-c}}\right] \times \left[c, \frac{c}{1-d\mathrm{e}^{-a}}\right]$$

is an invariant set for (1.1).

(ii) Let ϵ be an arbitrary positive number and (x_n, y_n) be an arbitrary solution of (1.1). We then consider the sets

$$I_1 = \left[a, \frac{a+\epsilon}{1-be^{-c}}\right], \qquad I_2 = \left[c, \frac{c+\epsilon}{1-de^{-a}}\right].$$
(2.8)

Then there exists an n_0 such that for all $n \ge n_0$

$$x_n \in I_1, \qquad y_n \in I_2. \tag{2.9}$$

Proof. (i) Let (x_n, y_n) be a solution of (1.1) with initial values x_{-1}, x_0, y_{-1}, y_0 such that

$$x_{-1}, x_0 \in \left[a, \frac{a}{1-be^{-c}}\right], \quad y_{-1}, y_0 \in \left[c, \frac{c}{1-de^{-a}}\right].$$
 (2.10)

Then from (1.1) and (2.10) we get

$$a \le x_1 = a + bx_{-1}e^{-y_0} \le a + \frac{ab}{1 - be^{-c}}e^{-c} = \frac{a}{1 - be^{-c}}$$

$$c \le y_1 = c + dy_{-1}e^{-x_0} \le c + \frac{cd}{1 - de^{-a}}e^{-a} = \frac{c}{1 - de^{-a}}$$

Then it follows by induction that

$$a \le x_n \le \frac{a}{1-be^{-c}}, \qquad c \le y_n \le \frac{c}{1-de^{-a}}, \quad n = 1, 2, \dots.$$

This completes the proof of statement (i).

(ii) Let (x_n, y_n) be an arbitrary solution of (1.1). Therefore, from Proposition 2.1 we get

$$0 < l_1 = \liminf_{n \to \infty} x_n, \qquad 0 < l_2 = \liminf_{n \to \infty} y_n,$$

$$L_1 = \limsup_{n \to \infty} x_n < \infty, \qquad L_2 = \limsup_{n \to \infty} y_n < \infty.$$
(2.11)

It follows from (1.1) and (2.11) that

$$L_1 \leq a + bL_1 e^{-l_2}, \qquad l_1 \geq a + bl_1 e^{-L_2}, \qquad L_2 \leq c + dL_2 e^{-l_1}, \qquad l_2 \geq c + dl_2 e^{-L_1}$$

which imply that

$$a \le L_1 \le \frac{a}{1 - be^{-c}}, \qquad c \le L_2 \le \frac{c}{1 - de^{-a}}.$$
 (2.12)

Thus from (1.1), we see that there exists an n_0 such that (2.9) holds true. This completes the proof of the proposition.

In the next two propositions we will study the asymptotic behavior of the positive solutions of (1.1). The next lemma is a slight modification of Theorem 1.16 of [24] and for readers convenience we state it without its proof.

Lemma 2.1. Let $f, g, f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+, g : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be continuous functions, $\mathbb{R}^+ = (0, \infty)$ and a_1, b_1, a_2, b_2 be positive numbers such that $a_1 < b_1, a_2 < b_2$. Suppose that

$$f:[a_1,b_1] \times [a_2,b_2] \to [a_1,b_1], \qquad g:[a_1,b_1] \times [a_2,b_2] \to [a_2,b_2].$$
(2.13)

In addition, assume that f(x, y) (resp. g(x, y)) is decreasing with respect to y (resp. x) for every x (resp. y) and increasing with respect to x (resp. y) for every y (resp. x). Finally suppose that if m, M, r, R are real numbers such that

$$M = f(M, r), \qquad m = f(m, R), \qquad R = g(m, R), \qquad r = g(M, r), \tag{2.14}$$

then m = M and r = R. Then the following system of difference equations

$$x_{n+1} = f(x_{n-1}, y_n), \qquad y_{n+1} = g(x_n, y_{n-1})$$
(2.15)

has a unique positive equilibrium (\bar{x}, \bar{y}) and every positive solution (x_n, y_n) of the system (2.15) which satisfies

$$x_{n_0} \in [a_1, b_1], \quad x_{n_0+1} \in [a_1, b_1], \quad y_{n_0} \in [a_2, b_2], \quad y_{n_0+1} \in [a_2, b_2], \quad n_0 \in \mathbb{N}$$

$$(2.16)$$

tends to the unique positive equilibrium of (2.15).

Proposition 2.3. Consider system (1.1) such that the following relations hold true:

If
$$c \geq a$$

$$b < e^{c} \frac{-a + \sqrt{a^{2} + 4}}{2}, \qquad d < e^{a} \min\left\{\frac{-c + \sqrt{c^{2} + 4}}{2}, \frac{c - \sqrt{c^{2} - a^{2}}}{a}\right\}$$
 (2.17)

and if $a \ge c$, then

$$d < e^{a} \frac{-c + \sqrt{c^{2} + 4}}{2}, \qquad b < e^{c} \min\left\{\frac{-a + \sqrt{a^{2} + 4}}{2}, \frac{a - \sqrt{a^{2} - c^{2}}}{c}\right\}.$$
(2.18)

Then system (1.1) has a unique positive equilibrium (\bar{x}, \bar{y}) and every positive solution of (1.1) tends to the unique positive equilibrium (\bar{x}, \bar{y}) as $n \to \infty$.

Proof. We consider the functions

$$f(x, y) = a + bxe^{-y}, \quad g(x, y) = c + dye^{-x}$$
(2.19)

where

$$x \in I_1, \quad y \in I_2, \tag{2.20}$$

 I_1, I_2 are defined in (2.8). Then from (2.17), (2.18)–(2.20), we see that for $x \in I_1, y \in I_2$

$$a \le f(x, y) \le a + b \frac{a + \epsilon}{1 - be^{-c}} e^{-c} = \frac{a + \epsilon be^{-c}}{1 - be^{-c}} < \frac{a + \epsilon}{1 - be^{-c}}$$

$$c \le g(x, y) \le c + d \frac{c + \epsilon}{1 - de^{-a}} e^{-a} = \frac{c + \epsilon de^{-a}}{1 - de^{-a}} < \frac{c + \epsilon}{1 - de^{-a}}$$

and so $f : I_1 \times I_2 \rightarrow I_1$, $g : I_1 \times I_2 \rightarrow I_2$. Let (x_n, y_n) be an arbitrary solution of (1.1). Therefore, as relations (2.17), (2.18) imply conditions (2.1), from Proposition 2.2 there exists an n_0 such that relations (2.9) hold true.

Let m, M, r, R be positive real numbers such that

$$M = a + bMe^{-r}, \quad m = a + bme^{-R}, \quad R = c + dRe^{-m}, \quad r = c + dre^{-M}.$$
 (2.21)

From (2.21) it follows that

$$r = \ln\left(\frac{bM}{M-a}\right), \qquad R = \ln\left(\frac{bm}{m-a}\right), \qquad m = \ln\left(\frac{dR}{R-c}\right), \qquad M = \ln\left(\frac{dr}{r-c}\right).$$
 (2.22)

Thus we see that relations (2.21) and (2.22) imply

$$(1 - be^{-r})\ln\left(\frac{dr}{r - c}\right) = a, \qquad (1 - be^{-R})\ln\left(\frac{dR}{R - c}\right) = a,$$

$$(1 - de^{-m})\ln\left(\frac{bm}{m - a}\right) = c, \qquad (1 - de^{-M})\ln\left(\frac{bM}{M - a}\right) = c.$$
(2.23)

We then consider the function

$$F(x) = (1 - de^{-x}) \ln\left(\frac{bx}{x - a}\right) - c.$$
 (2.24)

Let *z* be a solution of F(x) = 0. We claim that

$$F'(z) < 0.$$
 (2.25)

From (2.24) we see that

$$F'(z) = -\frac{a(1 - de^{-z})}{z(z - a)} + de^{-z} \ln\left(\frac{bz}{z - a}\right).$$
(2.26)

Since *z* satisfies equation F(x) = 0, then it follows that

$$\ln\left(\frac{bz}{z-a}\right) = \frac{c}{1-de^{-z}}.$$
(2.27)

Therefore, relations (2.26) and (2.27) imply that

$$F'(z) = -\frac{a(1 - de^{-z})}{z(z - a)} + \frac{cde^{-z}}{1 - de^{-z}}.$$
(2.28)

Using (2.28), to prove our claim (2.25), it suffices to prove that

$$H(z) - G(z) < 0,$$
 $H(z) = dcz(z - a),$ $G(z) = ae^{z}(1 - de^{-z})^{2}.$ (2.29)

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From (2.29) we get

$$H'(z) = dc(2z - a), \qquad G'(z) = -ad^2 e^{-z} + ae^z, \qquad H''(z) = 2dc, G''(z) = ad^2 e^{-z} + ae^z \qquad H'''(z) = 0, \qquad G'''(z) = -ad^2 e^{-z} + ae^z.$$
(2.30)

Now from (2.17), (2.18) and (2.30), we see that as z > a we have

$$H''(z) - G'''(z) < 0. (2.31)$$

Since z > a, we take

$$H''(z) - G''(z) < H''(a) - G''(a).$$
(2.32)

Using (2.30) we get

$$H''(a) - G''(a) = 2dc - ad^2 e^{-a} - ae^a = -e^{-a}(ad^2 - 2dce^a + ae^{2a}).$$
(2.33)

Moreover if $c \ge a$, then from (2.17) it follows that $0 < d < e^{a \frac{c - \sqrt{c^2 - a^2}}{a}}$ and we get

$$ad^2 - 2dce^a + ae^{2a} > 0. (2.34)$$

If $c \le a$ we can easily prove that (2.34) holds true. Then from (2.33) and (2.34) we get H''(a) - G''(a) < 0 and so from (2.32) it follows that

$$H''(z) - G''(z) < 0. (2.35)$$

Therefore from (2.35) and since z > a it follows

$$H'(z) - G'(z) < H'(a) - G'(a).$$
(2.36)

Hence using (2.30) we get

$$H'(a) - G'(a) = acd + ad^{2}e^{-a} - ae^{a} = ae^{-a}(d^{2} + dce^{a} - e^{2a}).$$
(2.37)

Now observe that from (2.17), (2.18) we have $0 < d < e^{a} \frac{-c + \sqrt{c^2 + 4}}{2}$ and so

$$d^2 + dce^a - e^{2a} < 0. (2.38)$$

Therefore relations (2.37) and (2.38) imply that H'(a) - G'(a) < 0 and so from (2.36) it follows that

$$H'(z) - G'(z) < 0. (2.39)$$

Hence from (2.39) and as z > a, we get

$$H(z) - G(z) < H(a) - G(a).$$
(2.40)

Now note that from (2.29) it is obvious that H(a) - G(a) < 0. Thus from (2.40), we get H(z) - G(z) < 0 which implies that (2.25) is true. Since (2.25) holds, it is known that there exists an ϵ such that for $x \in (z - \epsilon, z + \epsilon)$

$$F'(x) < 0.$$
 (2.41)

Therefore from (2.41) the function *F* is decreasing in the interval $(z - \epsilon, z + \epsilon)$. Suppose that *F* has roots greater than the root *z*. Let z_1 be the smallest root of *F* such that $z_1 > z$. From the argument above, we can show that there exists an ϵ_1 such that *F* is decreasing in the interval $(z_1 - \epsilon_1, z_1 + \epsilon_1)$. Since $F(z + \epsilon) < 0$, $F(z_1 - \epsilon_1) > 0$ and *F* is continuous, we see that *F* must have a root in the interval $(z + \epsilon, z_1 - \epsilon_1)$. This is clearly a contradiction since z_1 is the smallest root of *F* such that $z_1 > z$. Similarly we can prove that *F* has no solutions in (0, z). Therefore equation F(x) = 0 must have a unique solution. Hence from (2.23) and (2.24) *m*, *M* are the solutions of the equation F(x) = 0. Thus we see that m = M. Similarly if we set

$$G(x) = (1 - be^{-x}) \ln\left(\frac{dx}{x - c}\right) - a$$

.

and using (2.17), (2.18) we can show that equation G(x) = 0 has a unique solution. Also as r, R are the solutions of equation G(x) = 0, it follows that r = R. Therefore from Lemma 2.1 the proof of the proposition is complete. \Box

Proposition 2.4. *Consider system* (1.1) *and suppose that the constants a, b, c, d satisfy the following relations:*

$$b < \frac{e^{c}}{c+1}, \qquad d < e^{a} \min\left\{\frac{1-be^{-c}}{1-be^{-c}+a}, \frac{1-b(c+1)e^{-c}}{1-be^{-c}}\right\}.$$
 (2.42)

Then system (1.1) has unique positive equilibrium (\bar{x}, \bar{y}) such that

$$\bar{x} \in \left(a, \frac{a}{1-be^{-c}}\right), \qquad \bar{y} \in \left(c, \frac{c}{1-de^{-a}}\right).$$
(2.43)

Moreover every positive solution of (1.1) tends to the unique positive equilibrium (\bar{x}, \bar{y}) as $n \to \infty$.

Proof. First we prove that (1.1) has a unique positive equilibrium such that relations (2.43) hold. First we consider the following system of algebraic equations

$$x = a + bxe^{-y}, \quad y = c + dye^{-x}.$$
 (2.44)

Observe that system (2.44) is equivalent to the following system:

$$x = \frac{a}{1 - be^{-y}}, \qquad y = \frac{c}{1 - de^{-x}}.$$
 (2.45)

So we set

$$F(x) = \frac{a}{1 - be^{-f(x)}} - x, \qquad f(x) = \frac{c}{1 - de^{-x}}, \quad x \in \left(a, \frac{a}{1 - be^{-c}}\right).$$
(2.46)

Then from (2.42) and (2.46) we get

. . .

$$F(a) = \frac{abe^{-f(a)}}{1 - be^{-f(a)}} > 0,$$

$$F\left(\frac{a}{1 - be^{-c}}\right) = \frac{a}{1 - be^{-f\left(\frac{a}{1 - be^{-c}}\right)}} - \frac{a}{1 - be^{-c}}$$

$$= \frac{ab}{\left(1 - be^{-f\left(\frac{a}{1 - be^{-c}}\right)}\right)(1 - be^{-c})} \left(e^{-f\left(\frac{a}{1 - be^{-c}}\right)} - e^{-c}\right) < 0.$$
(2.47)

Therefore from (2.47) equation F(x) = 0 has a solution $\bar{x} \in \left(a, \frac{a}{1-be^{-c}}\right)$.

Now we will prove that \bar{x} is the unique solution of F(x) = 0. From (2.42) and (2.46) it follows that

$$F'(x) = \frac{abcde^{-f(x)-x}}{(1-be^{-f(x)})^2(1-de^{-x})^2} - 1 < \frac{abcde^{-c-a}}{(1-be^{-c})^2(1-de^{-a})^2} - 1.$$
(2.48)

Moreover from (2.42) we get,

$$be^{-c} + \frac{cb}{1 - de^{-a}}e^{-c} < 1, \qquad de^{-a} + \frac{ad}{1 - be^{-c}}e^{-a} < 1.$$
 (2.49)

Therefore from (2.49) it follows that

$$\frac{cb}{(1-de^{-a})(1-be^{-c})}e^{-c} < 1, \qquad \frac{ad}{(1-de^{-a})(1-be^{-c})}e^{-a} < 1.$$
(2.50)

Hence relations (2.48) and (2.50) imply that F'(x) < 0 which implies that equation F(x) = 0 has a unique solution $\bar{x} \in \left(a, \frac{a}{1-be^{-c}}\right)$. Then from (2.45) and (2.46) system (2.44) has a unique solution (\bar{x}, \bar{y}) such that (2.43) holds.

Let (x_n, y_n) be an arbitrary solution of (1.1). Using relations (2.42) and Proposition 2.1, we see that (2.11) hold which also imply that

$$l_1 \le \frac{a}{1 - be^{-l_2}}, \qquad l_1 \ge \frac{a}{1 - be^{-L_2}}, \qquad L_2 \le \frac{c}{1 - de^{-l_1}}, \qquad l_2 \ge \frac{c}{1 - de^{-L_1}}.$$
 (2.51)

From (2.51) we get

$$L_1 l_2 \leq \frac{a l_2}{1 - b e^{-l_2}}, \qquad l_1 L_2 \geq \frac{a L_2}{1 - b e^{-L_2}}, \qquad L_2 l_1 \leq \frac{c l_1}{1 - d e^{-l_1}}, \qquad l_2 L_1 \geq \frac{c L_1}{1 - d e^{-L_1}}$$

and so we see that

$$\frac{cL_1}{1 - de^{-L_1}} \le \frac{al_2}{1 - be^{-l_2}}, \qquad \frac{aL_2}{1 - be^{-L_2}} \le \frac{cl_1}{1 - de^{-l_1}}.$$
(2.52)

Now we consider the functions

$$f(x) = \frac{cx}{1 - de^{-x}}, \qquad g(y) = \frac{ay}{1 - be^{-y}}, \quad x \in \left(a, \frac{a}{1 - be^{-c}}\right), \ y \in \left(c, \frac{c}{1 - de^{-a}}\right).$$
(2.53)

Then from (2.53) it follows that

$$f'(x) = \frac{c(1 - de^{-x}(1 + x))}{(1 - de^{-x})^2}, \qquad g'(x) = \frac{a(1 - be^{-y}(1 + y))}{(1 - be^{-y})^2}.$$
(2.54)

From (2.49), consider $x \in \left(a, \frac{a}{1-be^{-c}}\right), y \in \left(c, \frac{c}{1-de^{-a}}\right)$

$$1 - de^{-x}(1+x) > 1 - de^{-a} \left(1 + \frac{a}{1 - be^{-c}} \right) > 0,$$

$$1 - be^{-y}(1+y) > 1 - be^{-c} \left(1 + \frac{c}{1 - de^{-a}} \right) > 0.$$
(2.55)

Therefore from (2.54) and (2.55) we see that

$$f'(x) > 0, \qquad g'(x) > 0, \quad x \in \left(a, \frac{a}{1 - be^{-c}}\right), \ y \in \left(c, \frac{c}{1 - de^{-a}}\right).$$

Hence, f, g are increasing functions and this, together with (2.52) implies that $l_1 = L_1$. Then, from (2.52) again, we see that $l_2 = L_2$. Therefore, this completes the proof of the proposition.

In the last proposition of this section, we will study the global asymptotic stability of the positive equilibrium of (1.1).

Proposition 2.5. Consider system (1.1) such that either (2.17) and (2.18) hold or (2.42) holds. Also suppose that the following relation holds true:

$$0 < be^{-c} + de^{-a} + bde^{-a-c} + \frac{abcde^{-a-c}}{(1 - de^{-a})(1 - be^{-c})} < 1.$$
(2.56)

Then the unique positive equilibrium (\bar{x}, \bar{y}) of (1.1) is globally asymptotically stable.

Proof. First we will prove that (\bar{x}, \bar{y}) is locally asymptotically stable. The linearized system of (1.1) about (\bar{x}, \bar{y}) is

$$\begin{aligned} x_{n+1} &= b e^{-y} x_{n-1} - b \bar{x} e^{-y} y_n \\ y_{n+1} &= d e^{-\bar{x}} y_{n-1} - d \bar{y} e^{-\bar{x}} x_n. \end{aligned}$$
(2.57)

We clearly see that system (2.57) is equivalent to the system

$$w_{n+1} = Aw_n, \quad A = \begin{pmatrix} 0 & \alpha & \beta & 0 \\ \gamma & 0 & 0 & \delta \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad w_n = \begin{pmatrix} x_n \\ y_n \\ x_{n-1} \\ y_{n-1} \end{pmatrix},$$
$$\alpha = -b\bar{x}e^{-\bar{y}}, \qquad \beta = be^{-\bar{y}} \quad \gamma = -d\bar{y}e^{-\bar{x}}, \qquad \delta = de^{-\bar{x}}$$

Then the characteristic equation of *A* is

$$\lambda^4 - (\beta + \delta + \alpha \gamma)\lambda^2 + \beta \delta = 0.$$

Since \bar{x} , \bar{y} satisfy (2.44) it is obvious that $\bar{x} > a$, $\bar{y} > c$. Hence, from (2.56) and as \bar{x} , \bar{y} satisfy (2.45) we get

$$|\beta| + |\delta| + |\beta\delta| + |\alpha\gamma| = be^{-\bar{y}} + de^{-\bar{x}} + be^{-\bar{y}}de^{-\bar{x}} + \frac{abcde^{-x-y}}{(1 - de^{-\bar{x}})(1 - be^{-\bar{y}})}$$

$$< be^{-c} + de^{-a} + bde^{-a-c} + \frac{abcde^{-a-c}}{(1 - de^{-a})(1 - be^{-c})} < 1.$$
(2.59)

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(2.58)

Therefore, from (2.59) and Remark 1.3.1 of [2] all the roots of Eq. (2.58) are of modulus less than 1 which implies that (\bar{x}, \bar{y}) is locally asymptotically stable. Using Proposition 2.3 (\bar{x}, \bar{y}) is globally asymptotically stable. This completes the proof of the proposition.

3. Existence of unbounded solutions of system (1.1)

It is our goal of this section to study the existence of unbounded solutions for (1.1).

Proposition 3.1. *Consider Eq.* (1.1). *Then the following statements are true:*

(i) Suppose that

$$b > e^c. \tag{3.1}$$

Then there exist solutions (x_n, y_n) of (1.1) such that

$$\lim_{n \to \infty} x_n = \infty, \qquad \lim_{n \to \infty} y_n = c.$$
(3.2)

(ii) Suppose that

$$d > e^a. \tag{3.3}$$

Then there exists a solution (x_n, y_n) of (1.1) such that

$$\lim_{n \to \infty} x_n = a, \qquad \lim_{n \to \infty} y_n = \infty.$$
(3.4)

(iii) Suppose that

$$b > e^c, \qquad d > e^a. \tag{3.5}$$

Then there exists a solution (x_n, y_n) of (1.1) such that either (3.2) or (3.4) holds true.

Proof. (i) Using (3.1), consider a solution (x_n, y_n) of (1.1) with initial values x_{-1}, x_0, y_{-1}, y_0 such that

$$x_{-1} > M, \quad x_0 > M, \quad y_0 < m, \quad y_{-1} < m, \quad M = \ln\left(\frac{dm}{m-c}\right), \ m = \ln b.$$
 (3.6)

Then from (1.1) and (3.6) we see that

 $y_1 = c + dy_{-1}e^{-x_0} < c + dme^{-M} = m, \qquad x_1 = a + bx_{-1}e^{-y_0} > a + bMe^{-m} > M.$

Therefore, it follows by induction that

$$y_n < m, \qquad x_n > M, \quad n = 1, 2, \dots$$
 (3.7)

Notice that from (1.1), (3.6) and (3.7) we get

$$x_{n+1} > a + bx_{n-1}e^{-m} = a + x_{n-1}, \quad n = 0, 1, \dots$$
(3.8)

Therefore from (3.8) it follows that

$$\lim_{n \to \infty} x_n = \infty. \tag{3.9}$$

In addition, from (1.1) and (3.7) we see that

 $c \le y_{n+1} = c + dy_{n-1}e^{-x_n} < c + dme^{-x_n}, \quad n = 0, 1, \dots$ (3.10)

Hence from (3.9) and (3.10) it follows that

$$\lim_{n \to \infty} y_n = c. \tag{3.11}$$

Thus from (3.9) and (3.11) the proof of statement (i) is completed.

(ii) Using (3.3) consider a solution (x_n, y_n) of (1.1) with initial values such that

$$x_{-1} < p, \quad x_0 < p, \quad y_0 > P, \quad y_{-1} > P, \quad P = \ln\left(\frac{bp}{p-a}\right), \ p = \ln d.$$
 (3.12)

Then applying (3.12) and using the ideas as above we can prove (3.4).

(iii) Using (3.5), it suffices to find a solution (x_n , y_n) with initial values satisfying either (3.6) or (3.12). Then (x_n , y_n) satisfies either (3.2) or (3.4). This completes the proof of the proposition. \Box

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