ZDM Mathematics Education (2008) 40:235–245
DOI 10.1007/s11858-008-0081-4

ORIGINAL ARTICLE

Comparative readings of the nature of the mathematical knowledge under construction in the classroom

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Accepted: 28 February 2008 / Published online: 11 March 2008

Abstract The present work reports on an attempt to empirically identify the epistemological status of mathematical knowledge interactively constituted in the classroom. To this purpose, three relevant theoretical constructs are employed in order to analyze two lessons provided by two secondary school teachers. The aim of these analyses was to enable a comparative reading of the nature of the mathematical knowledge under construction. The results show that each of these three perspectives allows access to specific features of this knowledge, which do not coincide. Moreover, when considered simultaneously, the three perspectives offer a rather informed view of the status of the knowledge at hand.

1 Introduction

Despite the considerable research interest shown in the last two decades in the study of the conditions under which the mathematical meaning is constructed in the classroom, the nature of the mathematical knowledge shaped within this context has attracted little attention. The reason for this rather limited research activity might be sought in the difficulty of defining coherently the exact status of the knowledge under consideration in didactical contexts. What do we mean by the term "school mathematics"? How does it relate to mathematics as a scientific discipline? Although the latter appears to play a decisive (but ambiguous) role in the determination of the former, the two types of knowledge present epistemological differences (Sierpinska & Lerman, 1996) with respect to their nature and structure. The epistemological status of the school mathematical knowledge cannot be deduced only from the scientific mathematical knowledge, but needs to be studied also in relation to the social contexts of the teaching and learning processes.

To this direction, the present work, which is concerned with the nature of the meaning emerging in the classroom and characterized as "mathematics", focuses on the classroom phenomena that determine this emergence. In particular, three relevant theoretical constructs are employed to investigate this nature, i.e., the idea of socio-mathematical norms (Yackel & Cobb, 1996), the notion of the epistemological triangle (Steinbring, 2006) and the analysis of the management of the epistemological features of mathematics (Kaldrimidou, Sakonidis & Tzekaki, 2000). These constructs are used to analyze two mathematics lessons, provided by two secondary school teachers, in an attempt to examine the different features of the mathematical knowledge shaped in the classroom that each one of these constructs allows to identify. Our claim is that the comparative and sometimes complementary use of different theoretical tools enables the sharpening of the analysis related to the mathematical status of this knowledge.

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2 Mathematics and school mathematics

All research in mathematics education deals with issues that have to do with mathematics: “mathematical meaning”, “mathematical activity”, “mathematical outcomes” (of students, teachers, communities, etc). However, the “mathematical” part in these expressions remains rather undefined and one could hardly justify why a meaning, an activity or an outcome can be characterized as “mathematical”.

It is generally admitted that school mathematics differs from science mathematics (Chevalard, 1985; Steinbring, 1998) because changes occur in the process of transformation from one to the other, both “externally” (from experts’ knowledge to knowledge for teaching) and “internally” (from knowledge for teaching to taught knowledge). In fact, there are researchers who consider school and science mathematics as completely different subject matters. For example, Sfard (1998, p. 494) argues that “although, mathematicians and mathematics education researchers deal with the same ‘subject matter’, the fact that they come from completely different paradigms is likely to make their views of mathematics incommensurable rather than merely different in some points”. Such a view raises a number of questions. For example, is it only the reconstruction of the mathematical knowledge for teaching purposes that changes it so substantially as to create new knowledge, different from the one that it comes from? What are the similarities and differences between school mathematics and science mathematics (e.g., with respect to concepts, procedures, structures, etc.)? When we teach mathematics, what do we refer to (the mathematics itself, a part of it, its nature, functioning, structures, etc.)?

The relation between a “teaching object” and the corresponding “mathematical object” is rather blurred. First, because mathematical objects and approaches adopted various forms and followed varied paths in the history of their development and the correspondence we are looking for is not so obvious. Second, as Ernest (2006, p. 73) points out, “most school mathematics topics are no longer a part of academic mathematics and thus figure in no contemporary academic textbooks”.

Whether one agrees or not with the aforementioned comments, it is evident that the study of the knowledge taught in the mathematics classroom requires certain clear criteria for what can be considered as “mathematics” and if this can be considered as such. As Godino and Batenero (1998, p. 177) argue, it has to be “based upon an analysis of the nature of mathematics and mathematical concepts.... Such epistemological analysis is essential in mathematics education for it would be very difficult to efficiently study the teaching and learning process of undefined and vague objects”.

It is widely accepted today that mathematical meanings or procedures are not simply “learnt” and “applied” by the students (e.g., Yackel, 2001; Steinbring, 1998), but are constructed, accepted or negotiated in the classroom (Voigt, 1994). Either as a personal or as a social construction, materialized in different contexts and in different ways (e.g., in action, in social interaction, etc.), school mathematics knowledge needs an agreement on whether what is personally or socially constructed is or is not mathematics. Moreover, the study of teaching and learning phenomena in the mathematics classroom and, in particular, the study of learners’ activity within the perspective of developing mathematical meanings need agreed detailed criteria with respect to the nature of the knowledge constructed.

In the search for these criteria, which will allow us to analyze the nature of the knowledge developed in the classroom, we attempt to exploit the above-mentioned theoretical approaches. Their choice was made on the basis that they all address the issue at hand in a direct and well-defined manner, each offering a coherent and detailed framework for its study. In addition, they focus on different aspects of the classroom meaning construction process in seeking to decide the mathematical status of this meaning, that is, social, individual and mathematical, respectively, thus enabling their comparative consideration. Although other perspectives could be seen as possible candidates for this enterprise, our search did not detect any which could be seen as satisfying the above-mentioned features equally well with these three particular approaches.

3 The theoretical approaches

As it has already been pointed out, the identification of what emerges as mathematics in the social context of classroom interaction is related to the epistemological status of the knowledge under construction.

To this direction, the first of these three approaches, the one of the sociomathematical norms (S-N), is concerned with the criteria according to which the mathematical status of the knowledge collectively constructed in the classroom is constituted.

The second approach, the one of the epistemological triangle (E-T), focuses on the mathematical nature of concepts interactively constructed in the classroom by the individual student with respect to their relational, abstract and general character within mathematics.

The third approach, the one of the classroom management of the epistemological features of mathematics (E-M), concentrates on the mathematical nature of both concepts and procedures interactively constructed in the classroom with respect to the role of these features in mathematics.

The basic elements of the above three approaches are briefly described below.

3.1 Sociomathematical norms (S-N)

The idea of the sociomathematical norms was conceived in order to analyze and describe the mathematical aspects of teachers' and students' activity in the mathematics classroom (Yackel & Cobb, 1996). These norms are collective criteria of values with respect to mathematical activities, which are constituted and continually regenerated and modified by the interactions taking place between the teacher and the pupils (Voigt, 1995). The sociomathematical norms are not predetermined, are context dependent and are established in all types of classrooms. The most common sociomathematical norms reported in the literature are specifically related to explanations, justifications and solutions. With respect to explanations and justifications, the main sociomathematical norm identified is related to "what counts as an acceptable mathematical explanation" (Yackel & Cobb, 1996). In particular, three categories of explanations have been identified:

- explanations as procedural descriptions;
- explanations as descriptions of actions on experientially real mathematical objects; and
- explanations as objects of reflection.

For example, focusing on a second grade classroom working on the addition of two-digit numbers (e.g., 12 + 13), Yackel and Cobb (1996, p. 469) interpret pupils' explanations with reference to the digits (1 plus 1 makes 2, 2 plus 3 makes 5) as procedural in nature, whereas explanations of the type "10 plus 10 makes 20, and 2 plus 3 are 5 more" as descriptions of actions on mathematical objects. They claim that the teacher's attitude to accept both solutions provided by the pupils but to promote the second one, thus legitimizing it, allows the establishment of a sociomathematical norm of "what counts as an acceptable mathematical explanation in the classroom". Negotiations about the adequacy and clarity of an explanation, which took place later in the year in the above class, are considered explanations as "objects of reflection". As a consequence, the related sociomathematical norms established in this class were, respectively: (a) explanations must describe actions on mathematical objects and should not constitute procedural instructions, and (b) explanations should aim at being understood by the pupils.

With reference to solutions, the related sociomathematical norms are concerned with "what is valued mathematically"; "what is a more sophisticated solution"; "what is an elegant mathematical solution" (Yackel & Cobb, 1996). Asking for a mathematically different solution (Yackel, Cobb & Wood, 1998) and evaluating the solutions using terms such as "insightful solution, simple solution, discoveries" (Voigt, 1995, p.198), the teacher helps the classroom to elaborate norms about what is mathematically efficient and/or what is mathematically different (Yackel & Cobb, 1996). For example, Voigt (1995, p. 197) reports on a teacher who accepted as correct the solving of the task of three additions with 9 (27 + 9, 37 + 9, 47 + 9) as three isolated problems. However, he characterized as 'an insightful' or as a 'discovery' the solving of the task by identifying the pattern of adding with 9 (that is, increase of the tens by one and decrease of the units by 1), a solution that the teacher seemed to consider cognitively as more demanding.

The teachers' role is crucial in establishing situations that highlight the importance of issues related to explanations, argumentation, justifications and solutions. As Yackel (1995, p. 160–161) points out "it is the teacher's responsibility to help students learn how to describe and talk about their mathematical thinking, to help them learn what constitutes an acceptable explanation.... Rather than taking the responsibility for judging this fits him or herself, the teacher can ask children if they understand and encourage them to ask questions and request clarification. In this way, the teacher contributes not only to children's developing understanding of what constitutes an acceptable explanation, but also to the interactive constitution of the obligation to listen to and try to make sense of the explanation attempts of others".

However, within the above perspective, the criteria related to the mathematical character of the knowledge under construction as well as the way these criteria affect their mathematical learning remain implicit. This is because these criteria are context-dependent and heavily subjected to personal interpretation. As individual pupils interact with the others, participate in collective negotiations of sociomathematical norms and try to adapt their activity to the classroom culture, they develop their personal interpretations of mathematical meanings and of values and beliefs about mathematical activity.

So far, the sociomathematical norms have been studied in the context of inquiry classrooms, the focus being mainly on the substantiation of the interactive constitution of these norms. Hence, they allow us to study how what is accepted as "mathematical" in the classroom is constructed, but they do not inform us whether the constructed knowledge is or is not mathematical in character.

3.2 The epistemological triangle (E-T)

Steinbring (2001, p. 211) focuses on the epistemological status of the mathematical knowledge, which is seen as interactively constructed by the students through working on concrete problems, being treated as exemplary cases "endowed with embodiments of mathematical structures". He advocates that this status can be identified through an epistemological analysis of the pupils' statements, that is, by examining whether the knowledge under construction reflected in these statements is oriented towards generalizing or it remains within the frame of the old, familiar factual knowledge or, finally, it is specific, partly situation-bound.

An analysis of this type can be achieved, according to Steinbring (2006, p. 136), through a relational structure called "the epistemological triangle", which allows us to consider the nature of the (invisible) mathematical knowledge shaped in the classroom by means of representing learner's construction of relations and structures during the relevant interaction. In particular, he suggests that the mathematical meaning of concepts emerges in the complex interplay between sign/symbol systems (for coding the knowledge) and reference contexts (for establishing the meaning of the knowledge), giving rise to the epistemological triangle below (Fig. 1).

The links between the corners of this triangle are seen as not explicitly defined but as forming a mutually supported, balanced system. As knowledge develops, the interpretations of sign systems and their corresponding reference contexts are modified. For example, considering the concept of probability, there is interplay between "fraction numerals" (sign system) and the "ideal die" (the reference context) in the early stages. Later, this interplay takes place between the "limit of the relative frequency" and the "statistical collectives" and even later between "stochastically dependent and independent structures" and "implicitly defined axioms", respectively (Steinbring, 2006, p. 138).

During the developmental process, the reference context gradually changes to a structural connection. For example, as the number concepts expands, concrete, empirical reference contexts (e.g., going up a staircase for adding) are increasingly substituted by others favoring relational aspects of the linkage between reference context and sign system, like diagrams, visualizing means (e.g., dots arranged in groups for adding) and even by other sign systems (e.g., number line for adding; Steinbring, 1997, pp. 54–55).

Based on the above, the production of mathematical meaning resulting from the interplay between reference context and sign system can be seen as a process via which possible meanings are transferred from a relatively familiar situation (the reference context) to a still unfamiliar sign system. Moreover, Steinbring (1998, p. 516) argues that as knowledge evolves, the roles of the reference context and the sign system can be exchanged, "leading to a situation where a familiar sign system serves as a reference context for another reference context, now conceived as a sign system with respect to some specific aspect".

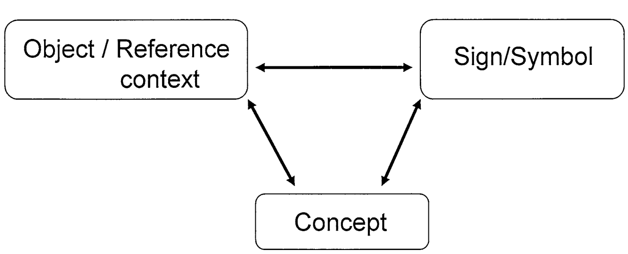

Fig. 1 Steinbring's epistemological triangle

In the course of classroom interaction, students have to actively construct likely relationships between signs/symbols and reference contexts. This personal construction turns to "official" in social negotiations with the teacher and the fellow students. The analysis of the classroom production of mathematical meaning within an epistemological perspective acknowledges that all mathematical knowledge is context-specific and therefore, the difference between scientific and school mathematics lies in the different types of reference contexts exploited in the course of development. Mathematical knowledge is theoretical in nature and thus abstract, relational and general. On the contrary, mathematics teaching, often aiming at obtaining a definite learning result, tends, in general, to provide empirical reference contexts and to avoid relational reference contexts for sign systems, thus promoting an empirical type of mathematical knowledge (Steinbring, 1998, pp. 523, 524):

> "…(which) accompanied by routinized interactive patterns of communication, such as the funnel pattern, changes meaningful mathematical understanding into conventionalized rules of algorithmic operations… (and) produce(s) a mythical interpretation of mathematical symbols that conflicts with the theoretical epistemology of mathematical knowledge because, in this way, students become accustomed to an artificially concrete understanding of mathematical concepts, and this produces epistemological obstacles to an understanding of the relational character of mathematical knowledge, that is unavoidable in later confrontations with new mathematical concepts".

The above perspective offers a way of looking at the nature of the mathematical knowledge under construction, particularly the knowledge related to concepts, leaving out, however, structural and functional elements of the mathematical activity. Such elements include processes like defining a mathematical object or reasoning mathematically.

3.3 Classroom management of the epistemological features of mathematics (E-M)

It has already been pointed out that the study of the nature of the knowledge under construction in the mathematics classroom is necessarily connected with the role and the function that concepts and procedures play in mathematics. In fact, it is generally accepted that learning mathematics means doing mathematics or, more generally, learning to think mathematically (Schoenfeld, 1992), which is unavoidably connected with functioning with the same "means" as mathematics does (Brousseau, 2006).

By what means does mathematics science operate? Among others, mathematics creates concepts, which are theoretical objects and uses definitions to identify and differentiate these objects from one another; it studies attributes and relations and uses theorems to present them. It also follows certain processes as means of management of objects and relationships and produces results or new objects. All these elements are of different nature and are used in an epistemologically different way. If there are aspects of this scientific activity to be developed in students' minds, these are not the formal procedures and rules but the mathematical ways of functioning and solving problems (Kaldrimidou, Sakonidis & Tzekaki, 2000).

Based on the above, we advocate that an analysis of classroom interaction with respect to the nature of the mathematical knowledge under construction should incorporate the study of the epistemological characteristics of the mathematics managed by both teachers and students: how the teacher and the students deal with the nature, the meaning and the definition of a concept, or how a theorem functions in solving, proving or validating procedures; in general, if and in what degree these important characteristics of the scientific activity are valued in the classroom (Kaldrimidou, Sakonidis & Tzekaki, 2007). More specifically, do teachers and pupils distinguish a specific case from a general one, are they in a position to define an object, and are the properties they use related to basic or implied properties? Do they solve a problem or explain a position by resorting to some basic attributes or relations or do they simply limit themselves to procedural negotiations? Do they analyze, compose or construct an object or do they simply recognize and describe it by visual means?

An analysis like the one suggested above allows the identification of a number of serious epistemological distortions evolving during the instructional process. For example, the teacher might replace a definition of a decimal fraction by a morphological description of the type "decimal fractions are written in the form of a decimal number recognised by the comma or decimal point"; or a concept, e.g., the area of a rectangle, by a procedure like "count the number of boxes" (Ikonomou, Kaldrimidou, Sakonidis & Tzekaki, 1999, p. 172). Also, he might reduce an argument to a property coming from a definition, e.g., "each angle is 45 degrees, because this is an isosceles, right-angled triangle" (Sakonidis, Tzekaki & Kaldrimidou, 2001, p.140). In other occasions, we might discover that a proving process is turned to measuring, or a solution process is equated to a course of operations of the type "do this and then that". Although certain researchers advocate that this passage from the procedural to the structural aspects of the mathematical concepts and processes is inevitable, some others suggest that this way of managing the mathematical objects and procedures distorts the nature of the meaning constructed in the classroom (Voigt, 1995).

Most of the current curricula support the need for the students to develop an awareness of the nature of mathematics, how it is created, used and communicated. Along this line, we argue that the nature of mathematical objects, like concepts, properties, relations and their role in the mathematical activity should constitute an important dimension of both teaching and learning processes, if students are to learn how to work mathematically. Otherwise, the activity developed in the mathematics classroom will bear none of the epistemological features characterising the mathematical processes. As a result, the means of carrying out a mathematical activity are likely to be mixed up, the methods of problem solving will constitute a typical, non-negotiable route to the solution, and the validation procedures (checking and confirming) will be submitted to the teacher's final approval.

Obviously, elements like definitions or theorems cannot always explicitly be presented to and identified by the students. However, the teacher needs to present, control and handle them in ways that respect the mathematical way of functioning and support students' understanding of the meaning and role of these features in the mathematical activity. To this direction, our research on the mathematical knowledge under construction in the classroom examines and compares with one another each discursive contribution made by both teachers and pupils in the course of their interaction in relation to the characteristics (a) assigned to it from a scientific mathematics point of view and (b) attributed to it in the context of the specific interaction (Kaldrimidou, Sakonidis & Tzekaki, 2000; Tzekaki, Kaldrimidou & Sakonidis, 2001; Kaldrimidou, Sakonidis & Tzekaki, 2003).

Summarizing, we argue that the above three perspectives allow us to look at different dimensions of the nature of the mathematical meaning constructed within the classroom, which could be described in general terms as follows: the socio-mathematical norms perspective allows us to identify what is "mathematical" by resorting to what is collectively accepted as such; the epistemological triangle approach permits us to examine whether what is interactively constructed by the individual student is "mathematical" by appealing to its relational character; finally, the classroom management of the epistemological features perspective offers us a way to look at whether what is formulated in the classroom is "mathematical" by considering its status within mathematics.

4 The study

The teaching episodes utilized in the present work are taken from lessons provided by two female secondary school teachers. These were "normal" teaching sessions to two different classes of ninth grade pupils (14–15 years old). Both teachers had a university degree in mathematics and more than 15 years of teaching experience.

The lessons are part of the data coming from a large project focusing on mathematics teaching in the 9 years of the Greek compulsory educational system.

For the purpose of the present work, we chose two episodes that could be discussed simultaneously from the point of view of all three theoretical perspectives: one concerned with the *concept* of algebraic fractions and the other with the solving *procedur*e of quadratic equations.

Our main intention was to investigate what each of them had to offer in relation to the nature of the mathematical knowledge produced. Methodologically speaking, we were not interested in analyzing systematically a large number of episodes or different mathematical contexts; thus, we focused on episodes, which would allow us to compare or combine the analytic benefits offered by each of the three perspectives.

5 Data analysis and discussion

For each episode under consideration, the analysis that follows concentrates first on the notion of the socio-mathematical norms (S-N), then on that of the epistemological triangle (E-T) and, finally, on the management of the epistemological features of mathematics (E-M).

5.1 Analysis of an episode from the first teacher's lesson

In the following episode, the class is working on algebraic fractions.

1. T(eacher): What does an algebraic fraction mean? Will you tell us Agy?
2. Agy: It is an expression, which has a variable as denominator.
3. T: Very good! It is an expression, which has a variable as denominator. Right? Now, Alexia, I want you to come to the board and write such a fractional algebraic expression. And Aphrodite, tell us exactly what we had (Alexia writes $1/x$).
4. T: Right! Harry, is x a variable?
5. Harry: Yes, it is.
6. T: Tell me, does it take all values?
7. Harry: Except 0.
8. T: Except 0, very nice! Why doesn't it take the value of 0, Christina?
9. Christina: Because the denominator becomes equal to 0.
10. T: I did not understand anything. So? Does it matter?
11. Christina: The denominator becomes equal to 0.
12. T: So what? Why does it matter?
13. Christina: We do not want it to be 0.
14. T: Why don't we want it to be 0?
15. Christina: There is no fraction with 0 as denominator
16. T: There is no fraction with 0 as denominator. How did we call this in primary school?
17. George: Division by 0.
18. T: Division by 0. Well done my dear!

From a mathematical point of view, the issue under negotiation in the classroom in this excerpt is the definition and the analysis of what constitutes a fractional algebraic expression. This requires the concept of variable as well as the criterion of the domain of a fractional algebraic expression.

The teacher, starting off by describing an algebraic fraction as an "expression with a variable in the denominator", asks for an example. Using the example provided by the students, e.g. $1/x$, she tries to elicit the reference to the condition "the denominator should be different from 0" for the rational algebraic expressions' domain. The students justify their answer with reference to fractions, "there is no fraction with 0 as denominator", while the teacher aims at an explanation of the type "division by 0 is impossible".

5.1.1 From a sociomathematical norms' perspective

Within this perspective, the negotiation between the teacher and the students in lines 1–8 appears to concern explanations as descriptions of mathematical objects (i.e., "It is an expression, which has a variable as denominator"). However, in lines 9–18, the teacher seeks for explanations, which can be interpreted as objects of reflection in terms of sociomathematical norms (i.e., "Why does it matter (that the denominator becomes equal to 0)?" or "Why don't we want it to be 0?"). This might be attributed to her wish to guide the students to the formulation "division by 0 is not allowed", which becomes evident by her overwhelming acceptance of this formulation, when provided by a student (lines 17 and 18).

The above analysis allows us to ascertain that what counts as an acceptable mathematical explanation is introduced by the teacher's questions (8, 10, 12, 16). The pupils' contributions appear as reactions to these questions, aiming at finding the formulation or the explanation expected by the teacher, which thus becomes "mathematically appropriate".

5.1.2 From an epistemological triangle's perspective

The sign system of the mathematical object under consideration is initially the fractional algebraic expression $1/x$. As the negotiation of its meaning goes on, the reference context changes: from fractional algebraic expressions (1–5) to fractional algebraic numbers (6–14), then to rational numbers (15, 16) and, finally, to an arithmetic operation (17, 18).

As the reference context changes from an algebraic expression to a rational number (by the student, first in line 2 and then in line 15), the corresponding relations that could legitimize these changes (i.e., an explanation of the type, "by giving values to the variables of an algebraic fraction, you get rational numbers") are not discussed explicitly and the whole changing enterprise remains implicit. Thus, it can be argued that the meaning of the related concept (algebraic fractions) remains rather blurred.

This analysis allows us to detect the interrelation between the concept and the corresponding reference context utilized, as well as the nature of the mathematical concept emerging in the classroom: the algebraic fraction is given the status of and is handled as a number.

5.1.3 From a management of epistemological elements' perspective

Considering the way in which the teacher poses the question at the beginning of this episode, one could claim that the mathematical object under consideration is the *definition* of an algebraic fraction.

However, examining the negotiation taking place within the management of the epistemological features framework, we cannot detect the identifying and discriminating role of a definition in the interaction.

As the teacher searches for the "right" explanation, three different mathematical objects (algebraic fractions, fractions, division) are introduced in the discussion and are implicitly interconnected. These objects are mainly presented in a morphological or procedural way and with no connection to definitions or properties (even informal, but accurate) that could help the new object (the rational algebraic expression) to be identified by the students.

Thus, there is not only a change of reference context in order to create a new piece of knowledge (the definition of an algebraic fraction), but also an interplay between different mathematical objects (fractions, equations, etc.) partly "defined" or even undefined, engaged in a rather blurred manner.

This analysis allows us to ascertain that the mathematical meaning of the algebraic fraction is apparently lost by the way the teacher handles the above-mentioned elements during the interaction, while the negotiation of the definition turns to a negotiation of a property (division by zero is not allowed).

Summarizing, the different elements that each of the above three analyses brings out with respect to the status of the mathematical knowledge under construction (i.e., the concept of algebraic fraction) are as follows: using indiscriminately procedural and morphological elements, the teacher demarcates what is accepted as "mathematical" (S-N); remaining in the same sign system ($1/x$), she changes reference systems without notifying, thus not facilitating the students to attend to relations established and to generalizations arising (E-T); the teacher turns a definition of a new object to the description of a property, entailing from the properties of other objects (E-M), without allowing the students to be led to the definition and generally to the clarification of the term that initiated the discussion.

5.2 **Analysis of an episode from the second teacher's lesson**

The topic of the lesson is the process of solving quadratic equations. The issue under consideration in the following episode is the solving process appropriate for each of three originally given equations: $x^2 - 2x = 0$, $x^2 - 4 = 0$ and $x^2 - 3x + 2 = 0$.

66. T: Children, let us look at some of these equations… I write down the equation $x^2 - 2x = 0$, another one, $x^2 - 4 = 0$ and a third one $x^2 - 3x + 2 = 0$. What do you notice in all these equations? There is an x, with what as an exponent?
67. Students. Two
68. T: When the highest exponent of the unknown is 2, as in our case, the equation is of second degree, because the highest exponent of a variable is called "the degree of that variable". Of what degree is x in this term?
69. Students: Second
70. T: So, the equation is of second degree in all three cases. Let us see how such an equation is solved in all three cases. I want to hear your opinion, children. How do you suggest we should solve the first equation? What shall we do on the left-hand side? Do you have an idea? How will we solve the equation [$x^2 - 2x = 0$], George?
71. George: We should separate the known from the unknown terms.
72. T: So, you suggest we separate. It cannot be done, because both terms are unknown. George expressed his opinion. Anyone else?

73. Margarita: Can't we factorize?
74. T: That's it, bravo! We will factorize the left hand side, and what will we have then, Margarita? Thus, what will happen to the left hand side? We will take out x as a common factor and what will we have inside, Harry?
[75–93] The class starts working on the second equation ($x^2 - 4 = 0$). In this case, the teacher accepts the two solving processes suggested by the students: factorization and separation of terms, both of which are mathematically applicable. Moreover, she emphasizes the separation of terms, which allows her to introduce new mathematical objects (negative square roots). However, a little later....
94. Kostas: Madam, in the first example, we had $x^2 - 2x = 0$, what if we do "x times x equals $2x$"? The x is canceled and then $x = 2$....
95. T: Watch it! Which x's will go?.... These x's are multiplied.... Priority of operations.... We first multiply....
96. Kostas: Madam, we will do $x^2 = 2x$.... $x.x = 2x$....
97. T: But you have a root! But you have a root! It is forbidden! Ok? You lose a root. Don't do this kind of cancelations, because you lose roots. All right? However, when we take out the common factor, we don't lose the root. $x = 0$, eh? We come up to $x = 0$. Don't do this kind of cancelations, because we lose a root.

5.2.1 *From a sociomathematical norms' perspective*

The analysis of the episode within this perspective allows the identification of the characteristics of the explanations provided as well as of the way in which the different solving processes suggested are judged and valued.

With respect to explanations, these are exclusively provided by the teacher (lines 68, 72, 95 and 97) and, as in the previous episode, are based either on procedural elements (lines 68 and 74) or on non-negotiable rules of procedures (lines 95 and 97). Thus, according to the S-N approach, the explanations and the justifications emerging as mathematical during this episode are mainly explanations as procedural (or morphological) descriptions.

As for the solving processes suggested, they are judged only as right or wrong. The teacher rewards the one she judges as right (line 74) and rejects the one she identifies as wrong or "dangerous" (lines 72 and 95–97).

One of the students (Kostas), at a later point of the lesson, returns to the first equation, $x^2 - 2x = 0$ (line 94), and again suggests separating terms as a solving process, which was earlier rejected by the teacher (lines 71 and 72), but in an intermediate phase was emphatically used by her as an alternative way to solve the second equation ($x^2 - 4 = 0$). It could be argued that Kostas simply adopts an implicitly established norm (i.e., the right processes are the ones that have been approved by the teacher) with accuracy. Reacting to this, the teacher provides an explanation, which reflects an attempt to rely on rules ("priority of operations", line 95). This could be seen as an explanation on object, which, however, eventually takes the form of an explanation based on result ("you lose a root", line 97). In fact, the final explanation constitutes a prohibition rule, with no mathematical value, which serves, however, a teaching target, i.e., to avoid errors (losing a root), thus ensuring a satisfying performance.

The points raised above indicate that the sociomathematical norms evolving in this class promote the idea that what is mathematically acceptable is not determined by an explicit and clear way, recognizable by the pupils, but by unidentifiable reasons and rules controlled by the teacher.

5.2.2 *From an epistemological triangle's perspective*

Considering the episode from within this framework, it could first be noted that the reference context (solving quadratic equations with two terms) and the sign system (algebraic expression of these equations) remain rather stable.

Some intangible changes of the reference context can also be identified. For example, when defining a quadratic equation (line 68), the teacher makes a reference to the second-degree algebraic expressions or polynomials through the use of the term "variable" instead of the term "unknown".

The lenses offered by this particular perspective allow the detection of another important aspect of the lesson. As it was pointed out earlier, one of the students (George) proposed a solving procedure, which the teacher rejected. Another student (Kostas) returned to this proposal, after a similar approach was applied by the teacher in solving the second equation ($x^2 - 4 = 0$) in the meantime (lines 94–96). The mathematical negotiation of this proposal would require the change of the reference context (to the management of algebraic expressions), since the cancelation demands division with divisor different from 0 (either $x = 0$ or $x \neq 0$, hence $x = 2$). This could allow a more intrinsic analysis and facilitate a generalization of the solving processes.

The teacher, however, rejects this in an authoritarian manner: "it cannot be done, because both terms are unknown" (line 72), "it is forbidden" (line 97) and she does not proceed to change the reference context. As a consequence, she limits the possible mathematical processes, thus in fact reducing the mathematical meaning. She does not only throw out a correct solving approach, but she also prevents the students from gaining a more general picture of how to solve quadratic equations.

Thus, the analysis of the episode in terms of the epistemological triangle allows us to notice that, in this case, the familiar sign system, that is, the solving of first-degree equations by separation of terms is utilized as a reference context for the solving of the second quadratic equation ($x^2 - 4 = 0$) but not of the first ($x^2 - 2x = 0$), which could be conceived as a similar sign system with respect to some specific aspects. What is more, the teacher completely declines to change the reference context, thus not allowing the students to see the two equations in terms of the same sign system.

5.2.3 *From a management of epistemological elements' perspective*

The first thing to notice within this perspective is that the definition of a quadratic equation is being turned to a morphological description (lines 66–68).

The most interesting aspect worth pointing out is, however, the way the teacher handles students' contribution concerning the solving procedures. In particular, she initially rejects a student's idea ("separation", line 71), simply by providing a mathematically vague descriptive explanation ("it cannot be done, because both terms are unknown", line 72), with no reference to properties or theorems. Later on, she accepts another student's proposal (factorization, line 74), offering procedural explanations, again with no reference to relevant properties or theorems (e.g., the distributive property of multiplication over subtraction, the conditions under which a product is equal to zero). Thus, the students are led to concentrate on solving quadratic equations based on algorithmic rather than structural elements of mathematics.

A final point that could be made in analyzing the episode within this perspective is related to a student's contribution (line 94). This intervention could be seen by the teacher as an opportunity to clarify what had been left blurred up to that point, that is, the attributes of algebraic expressions, which support the solving of equations and determine the most appropriate route to their solution. On the contrary, threatening the students with the danger of "losing roots", as earlier, she rejects a correct way of solving, without any mathematical justification.

Summarizing, the analysis of this episode within each of the three perspectives highlights, as in the first episode, different aspects of the nature of the mathematical knowledge under construction (i.e., solving procedures of quadratic equations).

The mathematically correct solution is determined by the teacher, who creates no opportunities for the comparison and the evaluation of the proposed processes by her or the students. Quite the opposite, adopting an authoritarian attitude, she reduces the mathematical meaning of the knowledge under construction and even distorts it by prohibiting correct mathematical processes (S-N).

Also, she rigidly handles the links between reference context and sign system, which would need to change in this case, in order for the mathematical process of solving an equation to be fully developed. Moreover, her main concern "to avoid errors" ("do not lose roots", lines 95 and 97) leads her to deny or/and to suggest approaches, even contradictory ones, to ensure that students may proceed to solving equations (E-T).

Finally, the teacher refuses to rely on properties and theorems, which substantiate the solving processes employed, leading the students essentially to function with procedural rules and not in a mathematically justified manner, that is, on the basis of properties and theorems (E-M). The latter would allow them to not only approach solving equations effectively, but also to become conscious of more general ways of functioning in their mathematical activity.

6 Concluding remarks

The idea of *sociomathematical norms* seems to offer an especially useful tool for analyzing classroom interactive patterns, specifically connected to mathematics. However, these interactive patterns concern almost exclusively socially constructed characteristics, ignoring other features, which also influence the relation of the knowledge built in the classroom to mathematics. For example, the S-N approach enables us to identify that the second teacher, and hence her class, accept that the solution process of separating terms is not permitted for the equation $x^2 - 2x = 0$, but does not provide us with the means to examine the relation of this acceptance to what is mathematically correct.

As a consequence, this perspective allows us to identify the criteria, which determine the mathematical status of the knowledge constructed in the classroom, but not the relation of that knowledge to mathematics. Hence, it can be argued that the corresponding analysis provides evidence concerning "how" something counts as mathematics and not on "why" or "if" something is mathematical.

The *epistemological triangle* offers a way to identify epistemological aspects of the mathematical knowledge under construction via focusing on its relational and "generalizable" nature (i.e., whether it remains concrete and context specific or can be generalized). Also, it allows attendance to the route followed by the mathematical content and its management by the teacher and the pupils through the succession of reference contexts and their relation to sign/symbol systems. For example, this approach helps in identifying that in the first episode, the rapid succession of reference contexts, from an algebraic expression to rational numbers and then to an operation, with the sign system remaining the same (algebraic symbols), moves away from the mathematical knowledge under consideration, drawing attention to a reduced value mathematical concept with a very concrete character.

Consequently, this second perspective allows us to examine epistemologically the ongoing development of knowledge related to the corresponding theoretical one. However, there are other elements of the mathematical activity, such as the way in which a concept is defined or operates in mathematics that this perspective does not take into consideration.

The *management of the epistemological elements* perspective explicitly focuses on these (epistemological) elements of the mathematical activity and especially on the nature, the meaning and the role of them in the classroom interactions. We argue that these elements constitute also an important dimension of the teaching and learning processes, if students are to learn how to work mathematically.

For example, the undifferentiated management of the various distinct mathematical objects in the first episode, algebraic expressions, fractions and division and the imposition of rules and properties without any explanation in the second episode do not help in highlighting the status of mathematical properties and relations. This manner of dealing with mathematical objects and their properties distorts their nature and role in mathematics, possibly leading students to difficulties in approaching the substance of the mathematical activity.

Concerning the teaching and learning situations that these three perspectives allow us to look at, we argue that the S-N perspective can be useful in examining inquiry classrooms, focusing on the mathematical character of the explanations, justifications and solving procedures established in them. When the instructional situation is oriented towards the development of a mathematical concept, the E-T approach appears to be more beneficial, examining the mathematical character of this development. Finally, the E-M can be fruitful in both cases, concentrating on whether the structure and organization of the mathematical knowledge constructed in the classroom is compatible to those of mathematics.

The points raised above, as a result of the comparative reading of the same teaching episodes, imply that each of the three theoretical approaches reveal different aspects related to the nature of the mathematical knowledge emerging in the classroom. Hence, they suggest that the parallel exploitation of these approaches can be especially valuable, as it offers a more integrated understanding of the parameters determining this nature. In particular, it allows us to become aware that it is influenced by all three of the following: (a) the kind of knowledge and how it becomes collectively accepted "as mathematical" in the classroom (S-N), (b) whether it is conceptually related in the individual student's mind via the appropriate interaction between reference and sign systems (E-T) and (c) how this knowledge is related to the corresponding mathematical entity (E-M).

Thus, reading the first episode comparatively, we saw that the accepted definiton of an algebraic expression was based on morphological and procedural characteristics encouraged by the teacher (S-N). These characteristics refer mainly to numerical fractions and it is doubtful whether they can help students approach the intended concept (theoretical knowledge) of algebraic fractions (E-T). Furthermore, this particular way of presentation differs substantially from a mathematical definition of algebraic fractions (E-M). Similarly, in the second episode, the analysis showed that the solving procedures of quadratic equations accepted as mathematically correct ended up to be unconnected procedures determined by the teacher (S-N). Also, the suggested ways of solving equations concerned distinct forms of equations, thus preventing students' orientation towards generalizing the solving of quadratic equations (E-T). Finally, neither properties nor theorems were ever utilized, eliminating the possibility of emergence of mathematically justified procedures (E-M).

Based on the above, it could be argued that each of the three perspectives, as it was used for the reading of the same teaching episode, allows us to identify different aspects of the knowledge construction process, which affect the nature of what emerges as "mathematical" knowledge within the classroom. In particular, these aspects concern the legitimization of this knowledge within the classroom community (S-N), its theoretical orientation (E-T) and, finally, its status within mathematics (E-M). Furthermore, they point out to the need of combining, rather than comparing, the results of the three readings initially planned. This combination offers a more global understanding of the meaning of the construction process taking place in the mathematics classroom, compared to that provided by each of them separately.

There is no doubt that the above enterprise is but an isolated and possibly rather uncertain attempt with respect to its outcome to explore a very demanding, but rather exciting issue, that of combining theoretical approaches (Lerman, 2006). However, the didactical phenomena occurring in the mathematics classroom as well as the mathematical knowledge emerging within it are very complicated in nature with respect to social, individual and epistemological aspects. Thus, such multiple approaches are indispensable, but they have to carefully incorporate the issues raised above, in order to fully identify the nature of the mathematical knowledge interactively constructed in the classroom context.

References

Brousseau, G. (2006). Mathematics, didactical engineering and observation. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th conference of PME*, Vol. 1 (pp. 3–18). Prague: Charles University.

Chevallard, Y. (1985). *La transposition didactique*. Grenoble: La Pensée Sauvage.

Ernest, P. (2006). A semiotic perspective of mathematical activity: the case of number. *Educational Studies in Mathematics*, *61*, 67–101.

Godino, J., & Batanero, C. (1998). Clarifying the meaning of mathematical objects as priority area for research in mathematics education. In A. Sierpinska, & J. Kilpatrick (Eds.), *Mathematics education as a research domain* (pp. 177–196). Dordrecht: Kluwer.

Ikonomou, A., Kaldrimidou, M., Sakonidis, C., & Tzekaki, M. (1999). Interaction in the mathematics classroom: Some epistemological aspects. In I. Scwank (Ed.), *Proceedings of first conference of European research in mathematics education*, Vol. I (pp. 168–181). Osnabrueck: Forschungsinstitut fuer Mathematik Didaktik.

Kaldrimidou, M., Sakonidis, H., & Tzekaki, M. (2000). Epistemological features in the mathematics classroom: algebra and geometry. In T. Nakahara, & M. Koyama (Eds.), *Proceedings of the 24th conference of PME*, Vol. 3 (pp. 111–119). Hiroshima: University of Hiroshima.

Kaldrimidou, M., Sakonidis, H., & Tzekaki, M. (2003). Teachers' intervention in students' mathematics work: A classification. In M.A. Mariotti (Ed.), *Proceedings of third conference of European research in mathematics education, thematic group 12.* Bellaria: CERME.

Kaldrimidou M., Sakonidis H., & Tzekaki, M. (2007). On the mathematical knowledge under construction in the classroom: A comparative study. In J.-H. Woo, H.-C. Lew, K.-S. Park, & D.-Y. Seo (Eds.), *Proceedings of the 31st conference of PME*, Vol. 3 (pp. 89–96). Seoul: The Korea Society of Educational Studies in Mathematics.

Lerman, S. (2006). Theories of mathematics education: Is plurality a problem? *Zentralblatt für Didaktik der Mathematik*, *38*(1), 8–13.

Sakonidis, H., Tzekaki, M., & Kaldrimidou, M. (2001). Mathematics teaching practices in transition: Some meaning construction issues. In M. van den Hauvel-Panhuizen (Ed.), *Proceedings of the 25th conference of PME*, Vol. 4 (pp. 137–144). Utrecht: Faculty of Mathematics and Computer Science, Freudenthal Institute, Utrecht University.

Schoenfeld, A. H. (1992). Learning to think mathematically; Problem solving, metacognition and sense making in mathematics. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334–370). New York: MacMillan.

Sfard, A. (1998). The many faces of mathematics: Do mathematicians and researchers in mathematics education speak about the same thing? In A. Sierpinska, & J. Kilpatrick (Eds.), *Mathematics education as a research domain* (pp. 491–512). Dordrecht: Kluwer.

Sierpinska, A., & Lerman, S. (1996). Epistemologies of mathematics and of mathematics education. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics education* (pp. 827–876). Dordrecht: Kluwer.

Steinbring, H. (1997). Epistemological investigation of classroom interaction in elementary mathematics teaching. *Educational Studies in Mathematics*, *32*(1), 49–92.

Steinbring, H. (1998). Epistemological constraints of mathematical knowledge in social learning settings. In: A. Sierpinska, & J. Kilpatrick (Eds.), *Mathematics education as a research domain* (pp. 513–526). Dordrecht: Kluwer.

Steinbring, H. (2001). Analyses of mathematical interaction in teaching processes. In M. van den Hauvel-Panhuizen (Ed.), *Proceedings of the 25th conference of PME*, Vol. 1 (pp. 211–217). Utrecht: Faculty of Mathematics and Computer Science, Freudenthal Institute, Utrecht University.

Steinbring, H. (2006). What makes a sign a mathematical sign?—An epistemological perspective on mathematical interaction. *Educational Studies in Mathematics*, *61*(1/2), 133–162.

Tzekaki, M., Kaldrimidou, M., & Sakonidis, H. (2001). Reflections on teachers' practices in dealing with pupils' mathematical error. In J. Novotna (Ed.), *Proceedings of second conference of European research in mathematics education* (pp. 322–332). Prague: Charles University.

Yackel, E. (1995). Children's talk in inquiry mathematics classroom. In P. Cobb, & H. Bauersfeld (Eds.), *The emergence of mathematical meaning* (pp. 131–162). Hillsdale: LEA.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation and autonomy in maths. *Journal for Research in Mathematics Education*, *27*, 458–477.

Yackel, E., Cobb, P., & Wood, T. (1998). The interactive constitution of mathematical meaning in one second grade classroom: an illustrative example. *The Journal of Mathematical Behaviour*, *17*(4), 469–488.

Yackel, E. (2001). Explanation, justification and argumentation in mathematics classroom. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th conference of PME*, Vol. 1 (pp. 9–24). Utrecht: Faculty of Mathematics and Computer Science, Freudenthal Institute, Utrecht University.

Voigt, J. (1994). Negotiation of mathematical meaning and learning mathematics. *Educational Studies in Mathematics*, *26*, 275–298.

Voigt, J. (1995). Thematic patterns of interaction and sociomathematical norms. In: P. Cobb, & H. Bauersfeld (Eds.), *The emergence of mathematical meaning* (pp. 163–201). Hillsdale: LEA.