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An Optimal Control Method for Real-Time Irrigation Scheduling

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In this paper a systematic methodology for making real-time irrigation decisions is presented. A physically based representation of the dynamics of the soil-crop-atmosphere system is used. The variables characterizing the crop and soil status are concurrently simulated with an integrated state space model. Soil moisture and salinity conditions, which synergistically control the plant water uptake, are obtained by using lumped parameter mass balance models for the root zone. Crop yield is predicted by explicitly modeling the plant growth processes, such as assimilation, respiration, and transpiration, which are driven by the climatic inputs. The control model is an analytical optimization method for multistage multidimensional sequential decision-making problems. It is suitable for systems with nonlinear dynamics and objective functions. The method is based on local iterative approximations of the nonlinear problem with a linear quadratic problem. This approach is evaluated in a series of case studies, where optimal irrigation schedules are obtained on an hourly basis over the growing season.

1. INTRODUCTION

Agricultural activities are important for the economies of most countries. The demand for agricultural products increases continuously as a result of growing populations, higher incomes, and new uses of traditional products. It is estimated that production will increase by expansion of arable land and, mainly, by intensified use and better management of the production factors. There is therefore an immediate need for efficient use of resources, in particular of water, in the production process.

Modern agriculture is a specialized and highly mechanized industry, in which solar energy is transformed to useful organic products. During the growing season the farmer makes important operational decisions which affect the final yield. Competitive and efficient agriculture requires such decisions to be made optimally. In the short run, maximization of a performance index is sought by exhaustive allocation of the limiting production factors, that is, water, land, and nutrients among different crops. The index is usually net benefits, although in some cases, when water is scarce, the objective may shift to maximizing produced crop weight per unit of water used. In the long run, objectives are related to societal issues such as welfare, malnutrition, deforestration and climatic change, environmental pollution, income redistribution and employment, soil conservation, and energy consumption. Even if optimal use of resources can lead to short-term gain in crop yield, the long-term effects can be immeasurable. Yet, in current practice the farmers act with the short-term objective of maximizing the benefits of the production process by prudent use of the available resources. This is also the philosophy of our study, which aims in introducing modern decision-making methodologies for water use in agriculture.

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2. THE IRRIGATION SCHEDULING PROBLEM: LITERATURE REVIEW

Irrigation in a broad sense is the human effort to control the soil-crop-atmosphere continuum. In particular, the irrigation scheduling problem is to determine the optimal timing, quantity, and quality of artificially supplied water to the soil in order to control the crop yield. Given that for six major U.S. grain crops the average annual yield is 20–35% of the record yield, there is much to be done to increase the solar energy conversion efficiency, which is directly related to the water transpired by the crop.

In current irrigation practice many farmers irrigate by the calendar, that is, in a predecided schedule based on experience, while others receive water at scheduled times and use it whether the crop needs it or not. As *Hillel* [1987] reports, when rule of thumb methods are not in use, irrigation scheduling techniques are classified as soil, plant, or climate based. In the first case, measurements of soil moisture and salinity are taken at a site. When these parameters reach certain critical values, irrigation is applied. In the second case, variables characterizing the status of the plant, such as leaf temperature, water content, and color, are measured or visually inspected. In some cases, functions of these parameters, called stress indices, have been used to indicate the necessity of irrigation. In the third case, measurements of climatic variables, such as total absorbed radiation, air temperature, relative humidity, and wind speed, are used to estimate evapotranspiration during a given time period. Irrigation is then applied whenever the estimated water loss exceeds a threshold value. The central idea in our research is to concurrently predict all variables characterizing the plant and soil status, using a model of the soil-crop-climate system, and then make more informed real-time irrigation decisions on the basis of modern optimal control techniques.

Two distinct approaches can be used to tackle the irrigation scheduling problem. The physically based (causal or physiological) approach emphasizes the need for modeling the processes resulting in crop growth and their relation to climate and soil. The statistical (regression or correlative) approach relies on extensive field experimentation and uses statistical analysis of field measurements. Several studies in the literature mix the two approaches in different proportions. For example, the work of *Cordova and Bras* [1981], *Rhenals and Bras* [1981], *Ramirez and Bras* [1985], *Gini* [1984], and *Bras and Seo* [1987] uses a lumped parameter soil column model, assuming uniform moisture and salinity in the root zone. The water status and the salt movement is modeled by mass balance equations for the unsaturated and saturated zones. The crop and soil system is forced by random climatic variables. The probability distribution of the evapotranspiration flux is derived. Then an empirical yield-transpiration model [*Stewart et al.*, 1977] is used to predict crop yield. According to this model, when actual transpiration fails to reach the climatically defined potential value, the potential crop yield is reduced. Since stress is more important at critical growth stages, weighting coefficients are used for each stage.

Introducing the net benefits as a performance index, that is, the value of the crop yield minus the cost of irrigation, *Bras and Cordova* [1981] solve the multistage decision problem using stochastic dynamic programming. They obtain tables which show different actions to be taken every week, depending on the soil water content and the water available for irrigation. *Bras and Seo* [1987] use similar system dynamics but employ extended linear quadratic Gaussian control ideas [*Georgakakos*, 1984] to find the optimal irrigation policy on a daily basis.

The work of *Protopapas and Bras* [1987, 1988] and *Protopapas* [1988] suggests an integrated model of the soil-crop-climate interactions, which is based on the underlying physiological and hydrological processes. Soil moisture and salinity profiles are predicted solving the distributed parameter governing equations for flow and transport in the unsaturated zone. Crop yield is predicted by explicitly modeling the plant growth processes, such as assimilation, respiration, and transpiration. The model is developed in an analytical state space form, that is, as a set of nonlinear equations for the propagation of the plant and soil state variables. As will be seen in the following sections, this formulation is convenient for implementation of real-time optimal control methods. The thrust of this paper is that it solves the irrigation scheduling problem using such a physical system representation. As in the aforementioned studies, flow and transport dynamics are represented by simplified lumped parameter models. The results of the current formulation demonstrate the feasibility and validity of the approach.

This paper includes four additional sections. Section 3 presents the system model and formulates the control problem. Section 4 discusses the control algorithm and provides a step-by-step description of its implementation. The control scheme is tested in several case studies in section 5, and section 6 summarizes our conclusions and future research directions.

3. THE SOIL-CROP-CLIMATE MODEL

Assuming availability of nutrients, the processes resulting in crop growth are summarized as follows. The climatic inputs act on the canopy of the crop, resulting in photosynthesis and transpiration. The products of photosynthesis are the reserves of the plant and are utilized for maintenance and growth of the shoot and root biomass. The transpiration demand is met by water uptake through the root system at a rate which largely depends on the moisture and salinity conditions in the soil. By adjusting its water content (or equivalently its water potential) the plant equates demand and uptake and at the same time regulates the partitioning of biomass in shoot and root. *Protopapas and Bras* [1987, 1988] and *Protopapas* [1988] have quantified and mathematically expressed the above concepts, following ideas and data reported by *de Wit et al.* [1978]. In the sequel we propose a simplified version of the state space soil-crop-climate model, which does not consider vertical distribution of the root system, the soil moisture, and the salinity over depth.

Let us introduce a vector of state variables

$$x_\kappa = [\psi_p \ \mathrm{RES} \ W_S W_R \psi c]^T$$

where x_κ is the six-dimensional state vector at time κ, ψ_p is the plant water potential (in bars) characterizing the water status of the plant, RES is the weight of reserves (in kilograms per hectare), W_S and W_R are the weight of shoot and root (both in kilograms per hectare), ψ is the soil matric potential (in centimeters) characterizing the water content of the soil, and c is the soil salinity (in milligrams per liter). An important variable used in the model is the effective soil potential ψ_s, which combines the matric potential ψ related to water content and the osmotic potential related to soil salinity. Less negative ψ_s values indicate wet soil with low salinity, while more negative ψ_s values correspond to the opposite conditions.

Let us further define a vector of climatic inputs

$$\xi_\kappa = [R_n R_v T_a T_d T_s u r]^T$$

where ξ_κ is the seven-dimensional input vector at time κ; R_n and R_v are the net absorbed solar radiation and the photosynthetically active radiation, respectively (in Joules per square meter per second); T_a, T_d, and T_s are the air temperature, the dew point temperature, and the soil temperature, respectively (in degrees Celsius); u is the wind speed (in meters per second); and r is the precipitation rate (in centimeters per day).

Let us also define a control vector (scalar in our problem)

$$u_\kappa = i$$

where u_κ is the control variable at time κ, defined as the irrigation rate i (in centimeters per day). In this study we assume that it is possible to irrigate at any rate less than the saturated hydraulic conductivity of the soil and that all supplied water infiltrates locally, that is, no ponding and runoff occurs.

The state space formulation of the soil-crop-climate model is given in thorough detail by *Protopapas and Bras* [1988]. The simplified version with lumped description of the flow and transport dynamics used in this paper is presented in the appendix. Overall, the formulation explicitly relates the state vector at time ($\kappa + 1$) to the state, input, and control vectors at time κ through a nonlinear, time-varying function:

$$x_{\kappa+1} = f_\kappa(x_\kappa, \xi_\kappa, u_\kappa) \qquad (1)$$

In the deterministic case the input vectors ξ_κ are simply known a priori and do not enter the analysis any further. In Figures 1 and 2 we demonstrate two example cases, where the model simulates the response of the plant under favorable ("wet") conditions and unfavorable ("dry") conditions, corresponding to constant soil potential of −500 and

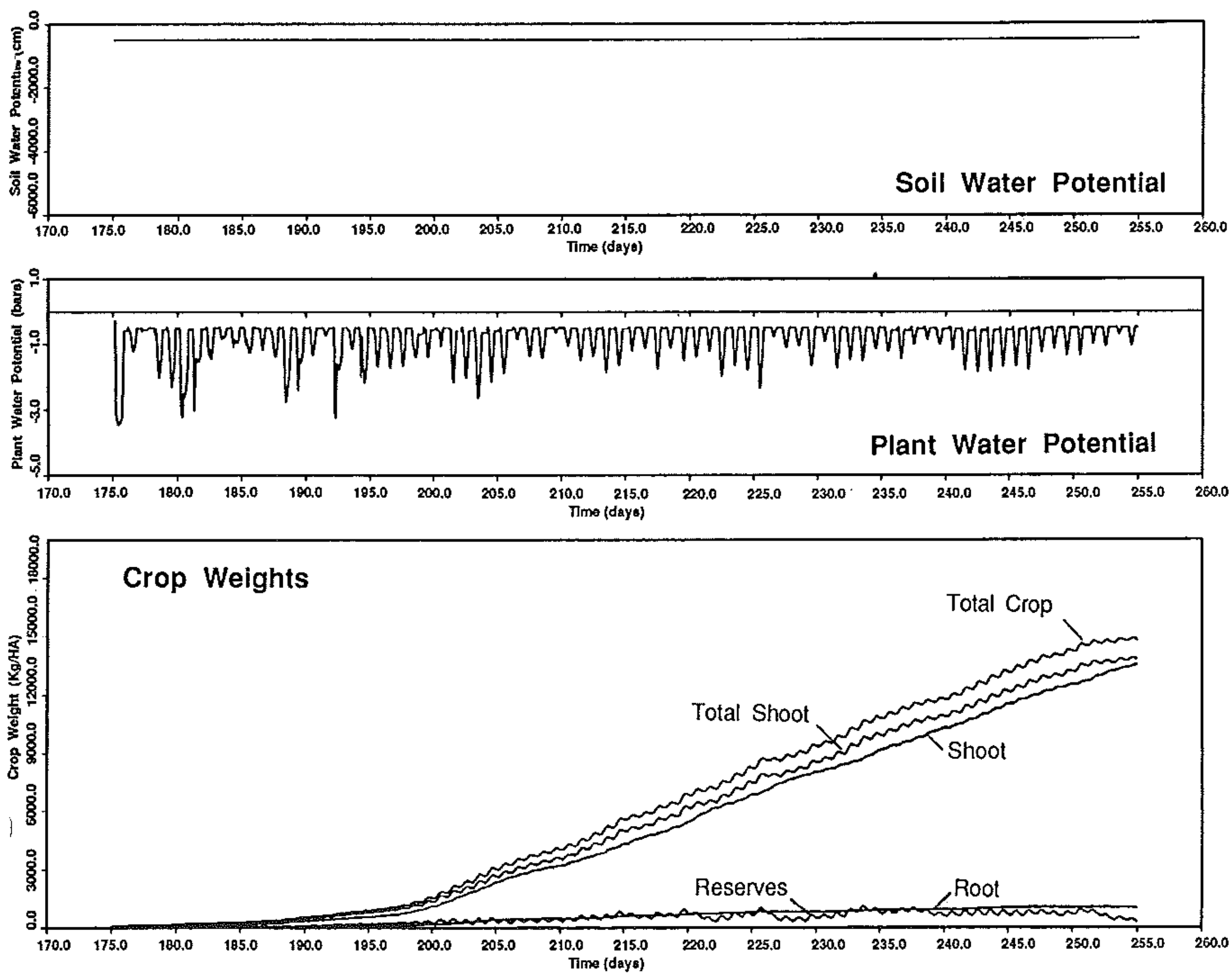

Fig. 1. Plant growth under wet soil conditions (ψ_s = −500 cm).

−2500 cm, respectively (top part). In the first case the plant water potential is close to zero over most of the simulated period, showing a diurnal pattern with a minimum value around noon (middle part). Since the plant does not suffer water stress, most of the produced biomass is allocated to shoot growth. After an initial period of slow growth the shoot weight increases almost linearly with time (lower part). In the second case the plant water potential drops to more negative values and the crop is under continuous water stress. The limited photosynthetic products are now distributed in favor of root growth. These simulations are for a maize crop, for which the parameters of the model are available. For more simulations and discussion see *Protopapas and Bras* [1988].

Clearly, maintaining wet soil conditions guarantees higher crop yields, but it also requires higher irrigation costs. A rational irrigation scheduling plan should optimally balance this trade-off, especially when water is in short supply. This can be formulated as a real-time optimal control problem, which seeks a sequence of controls u_κ, $\kappa = 0, 1, \cdots, N-1$, which minimizes the following performance index:

$$J = g_N(x_N) + \sum_{\kappa=0}^{N-1} g_\kappa(x_\kappa, u_\kappa) \qquad (2)$$

where J is the objective function or total net benefits from applying the policy $\{u_0, u_1, \cdots, u_{N-1}\}$, N is the number of time steps, and $g_\kappa(x_\kappa, u_\kappa)$ is a cost function associated with the state and control vectors at each time step κ. As will be seen in the application section, various $g_\kappa(\ )$ and $g_N(\ )$ functions can be adopted for the irrigation scheduling problem.

In addition to the state equation (1), this optimization is subject to a number of constraints on the state and control variables. For example, the plant and soil water potential cannot be positive, a minimum crop yield may need to be guaranteed, a maximum soil salinity at the end of the growing season may not be exceeded, and irrigation rates cannot be negative or exceed certain availability levels. Such constraints can be modeled as lower and upper bounds on the respective state and control variables:

$$x_\kappa^{\min} \le x_\kappa \le x_\kappa^{\max} \qquad (3a)$$

$$u_\kappa^{\min} \le u_\kappa \le u_\kappa^{\max} \qquad \kappa = 0, 1, \cdots, N \qquad (3b)$$

where the inequality (3*a*) holds componentwise.

The optimization problem summarized by (1), (2), (3*a*), and (3*b*) can be solved by general nonlinear programming methods [*Luenberger*, 1973]. However, given the dynamical

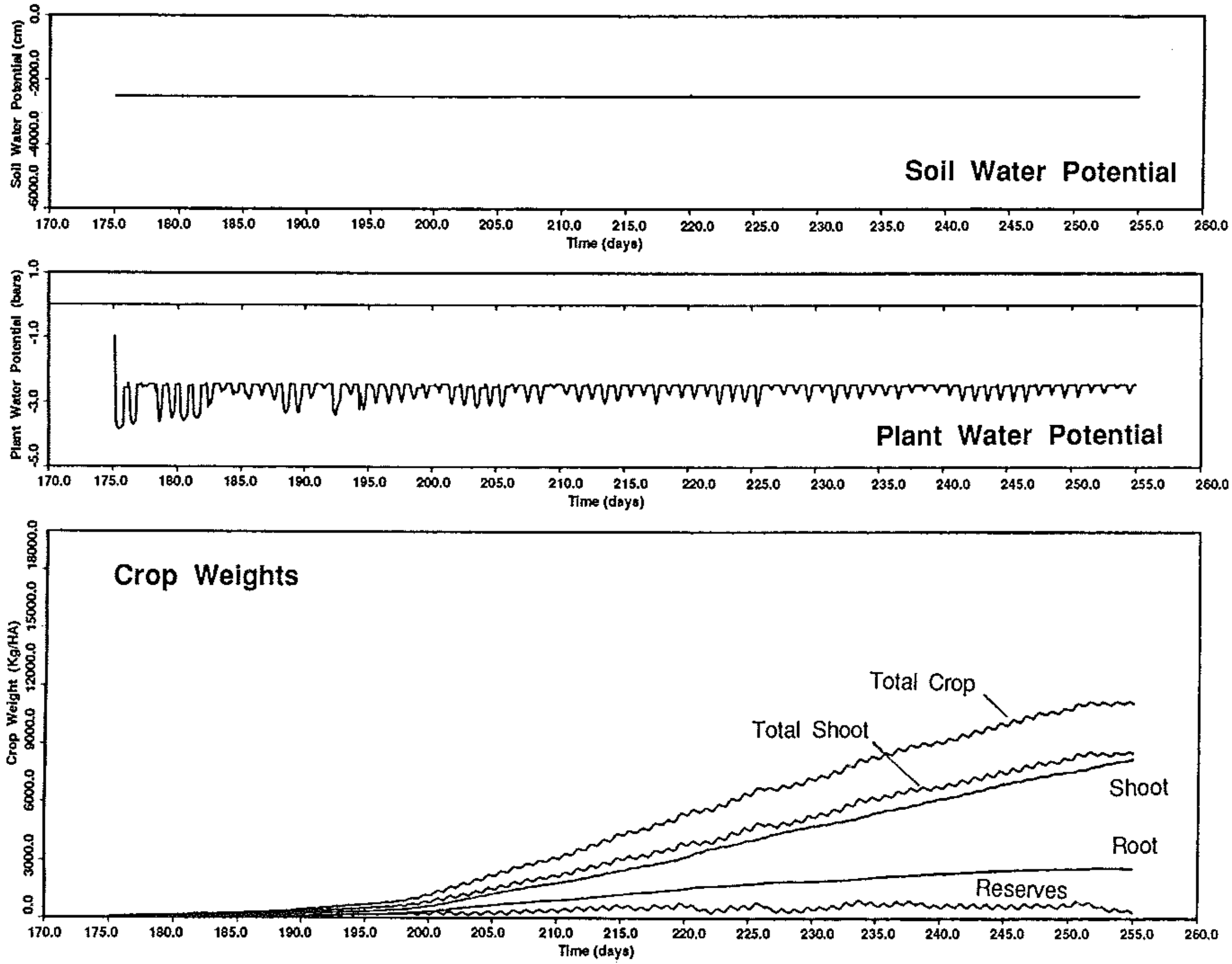

Fig. 2. Plant growth under dry soil conditions ($\psi_s = -2500$ cm).

process (1) and the time separability of the performance index (2), the solution process can greatly benefit from using optimal control methods [*Bertsekas*, 1987].

The above-stated problem is valid when the input vector κ_κ, $\kappa = 0, 1, \cdots, N-1$, is a known deterministic sequence. If ξ_κ is presumed random, then the state sequence generated by equation (1) and the performance index (2) will also be random. (In fact, this will be the case even if the control sequence u_κ, $\kappa = 0, 1, \cdots, N-1$, has a priori been decided.) The stochastic version of the irrigation scheduling problem will seek to minimize the expected value of the performance index (2) subject to the state dynamics (1) and constraints (3*a*) and (3*b*). Given the random nature of the state variables, constraints (3*a*) would have to be restated as follows:

$$\text{Prob}\,[x_\kappa^{\min} > x_\kappa] \le \gamma_\kappa^{\min}$$
$$\text{Prob}\,[x_\kappa > x_\kappa^{\max}] \le \gamma_\kappa^{\max} \qquad \kappa = 1, \cdots, N \tag{3c}$$

where $\gamma_\kappa^{\min}$ and $\gamma_\kappa^{\max}$ are probabilistic tolerance levels to be specified by the decision maker.

In this paper our focus is on the deterministic irrigation scheduling problem. The stochastic formulation is also amenable to optimal control methods but will be studied separately.

4. THE LINEAR QUADRATIC CONTROL METHOD

The control model is a generalized gradient method [*Luenberger*, 1973] which iterates using the Newton optimization direction. Generally, when minimizing a scalar real-valued function $f(x)$ with respect to the n-dimensional vector x, a generalized gradient method "moves" from point x_i to point x_{i+1} according to the following iteration:

$$x_{i+1} = x_i + \alpha_i d_i \tag{4}$$

where d_i is the descent direction (vector), and α_i represents the stepsize (scalar). Concerning the direction, the best, yet not always practical, choice is to use the Newton's direction:

$$d_i = -\left[\frac{\partial^2 f(x_i)}{\partial x^2}\right]^{-1} \frac{\partial f(x_i)}{\partial x} \tag{5}$$

which is obtained as the product of the inverse hessian and the gradient. At each iteration the Newton's direction is specified using first- and second-order derivative information concerning the shape of $f(x)$, this being its strength and weakness. The additional information can realize a faster convergence rate, yet it also requires heavier computational effort. In relation to the irrigation scheduling problem, the vector x includes all control variables at all times $\{u_\kappa, \kappa = 0, 1, \cdots, N-1\}$, and straightforward hessian computation is not recommended. However, the Newton's direction can be efficiently determined in an indirect manner, approximating the nonlinear problem, with a linear quadratic problem.

Consider a first-order Taylor's series expansion of the system dynamics (1) around nominal control and (corresponding) state trajectories:

$$\delta x_{\kappa+1} = A_\kappa \delta x_\kappa + B_\kappa \delta u_\kappa \qquad \delta x_0 = 0.0 \tag{6}$$

and (2) a second-order Taylor's series expansion of the cost terms $g_\kappa(x_\kappa, u_\kappa)$, $\kappa = 0, 1, \cdots, N-1$ and $g_N(x_N)$ in the objective function (2) also around the nominal values of the state and control vectors:

$$\tilde{g}_\kappa(\delta x_\kappa, \delta u_\kappa) = g_\kappa(\hat{x}_\kappa, \hat{u}_\kappa) + \tfrac{1}{2}\, \delta x_\kappa^T Q_\kappa \delta x_\kappa + \tfrac{1}{2} R_\kappa \delta u_\kappa^2 + \delta u_\kappa M_\kappa \delta x_\kappa + a_\kappa^T \delta x_\kappa + b_\kappa^T \delta u_\kappa \tag{7}$$

$$\tilde{g}_N(\delta x_N) = g_N(\hat{x}_N) + \tfrac{1}{2}\, \delta x_N^T Q_N \delta x_N + a_N^T \delta x_N \tag{8}$$

where we define

$$\delta x_\kappa = x_\kappa - \hat{x}_\kappa \qquad \delta u_\kappa = u_\kappa - \hat{u}_\kappa$$

with $\hat{u}_\kappa$ a nominal control trajectory and $\hat{x}_\kappa$ the corresponding state trajectory using the nonlinear system dynamics. The matrices in equations (6), (7), and (8) are defined from

$$A_\kappa = [\nabla_{x_\kappa} f_\kappa(\hat{x}_\kappa, \hat{u}_\kappa)]^T \qquad B_\kappa = [\nabla_{u_\kappa} f_\kappa(\hat{x}_\kappa, \hat{u}_\kappa)]^T$$

[6 × 6 matrix] [6 × 1 vector]

$$Q_\kappa = [\nabla^2_{x_\kappa x_\kappa} g_\kappa(\hat{x}_\kappa, \hat{u}_\kappa)] \qquad Q_N = [\nabla^2_{x_N x_N} g_N(\hat{x}_N)]$$

[6 × 6 matrix] [6 × 6 matrix]

$$R_\kappa = [\nabla^2_{u_\kappa u_\kappa} g_\kappa(\hat{x}_\kappa, \hat{u}_\kappa)] \qquad M_\kappa = [\nabla^2_{x_\kappa u_\kappa} g_\kappa(\hat{x}_\kappa, \hat{u}_\kappa)]$$

[scalar] [1 × 6 vector]

$$a_\kappa = \nabla_{x_\kappa} g_\kappa(\hat{x}_\kappa, \hat{u}_\kappa) \qquad a_N = \nabla_{x_\kappa} g_N(\hat{x}_N)$$

[6 × 1 vector] [6 × 1 vector]

$$b_\kappa = \nabla_{u_\kappa} g_\kappa(\hat{x}_\kappa, \hat{u}_\kappa)$$

[scalar]

where the symbols ∇ and ∇^2 represent the gradient vector and the hessian matrix, respectively.

Furthermore, consider the following control problem:

$$\text{Minimize } J = \tilde{g}_N(\delta x_N) + \sum_{\kappa=0}^{N-1} \tilde{g}_\kappa(\delta x_\kappa, \delta u_\kappa) \tag{9}$$

subject to the linear dynamics (6). It can be shown [*Bertsekas*, 1982] that the Newton direction for the nonlinear control problem with performance index (2) and state dynamics (1) can be obtained by solving the above stated problem. The advantage is that the second problem has linear dynamics and a quadratic performance index, and its solution can be derived analytically. The derivation is presented next in a step-by-step format:

1. Let $\{\hat{u}_\kappa, \kappa = 0, 1, \cdots, N-1\}$ be a nominal control sequence. Compute the associated nominal state sequence $\{\hat{x}_\kappa, \kappa = 1, \cdots, N\}$ through equation (1). Derive the matrices of the system dynamics A_κ and B_κ and the matrices or vectors of the cost terms Q_κ, Q_N, R_κ, M_κ, a_κ, a_N, and b_κ, as defined by the Taylor's series expansion of the nonlinear problem.

2. Determine the positive, semidefinite, [6 × 6]-dimensional matrices $\{K_\kappa, \kappa = 0, 1, \cdots, N\}$ and the [6 × 1]-dimensional vectors $\{\lambda_\kappa, \kappa = 0, 1, \cdots, N\}$ by performing the following recursive operations:

$$K_N = Q_N \tag{10}$$

$$K_\kappa = Q_\kappa + A_\kappa^T K_{\kappa+1} A_\kappa - (B_\kappa^T K_{\kappa+1} A_\kappa + M_\kappa)^T \cdot (R_\kappa + B_\kappa^T K_{\kappa+1} B_\kappa)^{-1} (B_\kappa^T K_{\kappa+1} A_\kappa + M_\kappa) \tag{11}$$

$$\kappa = N-1, N-2, \cdots, 1$$

$$\lambda_N = a_N \tag{12}$$

$$\lambda_\kappa = a_\kappa + A_\kappa^T \lambda_{\kappa+1} - (B_\kappa^T K_{\kappa+1} A_\kappa + M_\kappa)^T \cdot (R_\kappa + B_\kappa^T K_{\kappa+1} B_\kappa)^{-1} (b_\kappa + B_\kappa^T \lambda_{\kappa+1}) \tag{13}$$

$$\kappa = N-1, N-2, \cdots, 1$$

3. Compute the Newton minimization direction $\{d_\kappa, \kappa = 0, 1, \cdots, N-1\}$ as follows:

$$d_\kappa = -D_\kappa [L_\kappa \delta x_\kappa + \Lambda_\kappa] \qquad \kappa = 0, 1, \cdots, N-1 \tag{14}$$

where

$$D_\kappa = [B_\kappa^T K_{\kappa+1} B_\kappa + R_\kappa]^{-1} \tag{15}$$

$$L_\kappa = B_\kappa^T K_{\kappa+1} A_\kappa + M_\kappa \tag{16}$$

$$\Lambda_\kappa = B_\kappa^T \lambda_{\kappa+1} + b_\kappa \tag{17}$$

$$\delta x_{\kappa+1} = A_\kappa \delta x_\kappa + B_\kappa d_\kappa \qquad \kappa = 0, 1, \cdots, N-1$$

$$\delta x_0 = 0 \tag{18}$$

Essentially, computation of the direction d_κ requires two forward time recursions (steps 1 and 3) and one backward time recursion (step 2).

4. The second factor influencing the success of a minimization method is the stepsize selection rule. Among the available options are the minimization rule, the limited minimization rule, the Goldstein rule, the Armijo rule, and many others. The irrigation scheduling control model uses the Armijo stepsize selection rule because it is easily implemented and because it conveniently generalizes to problems with constraints. This procedure is as follows [*Bertsekas*, 1982]:

Let β and σ be scalars satisfying $0 < \beta < 1$ and $0 < \sigma < 0.5$. If $\{\hat{u}, = 0, 1, \cdots, N-1\}$ denotes the nominal control sequence and $\{d_\kappa, \kappa = 0, 1, \cdots, N-1\}$ represents the minimization direction, the stepsize α is obtained from $\alpha = \beta^m$, where m is the smallest nonnegative integer for which

$$J^{\text{old}} - J^{\text{new}} \geq -\sigma \beta^m [\partial J^{\text{old}} / \partial u_\kappa]^T d_\kappa \tag{19}$$

In the above inequality, J^{old} is the value of the performance index under the nominal control and state sequences while

J^{new} is its value when the controls are set equal to $u_\kappa = \hat{u}_\kappa + \beta^m d_\kappa$, $\kappa = 0, 1, \cdots, N-1$. The computation of the gradient vectors $\partial J^{\text{old}}/\partial u_\kappa$, $\kappa = 0, 1, \cdots, N-1$, can be obtained by the following backward time recursion:

$$p_N = a_N \tag{20}$$

$$p_\kappa = a_\kappa + A_\kappa^T p_{\kappa+1} \tag{21}$$

$$\partial J^{\text{old}}/\partial u_\kappa = R_\kappa + B_\kappa^T p_{\kappa+1} \qquad \kappa = N-1, N-2, \cdots, 0 \tag{22}$$

Starting with $m = 0$, the procedure is repeated for larger values of m until criterion (19) is passed.

5. After the specification of the stepsize α which satisfies criterion (19) the new control sequence is obtained from

$$u_\kappa = \hat{u}_\kappa + \alpha d_\kappa \qquad \kappa = 0, 1, \cdots, N-1 \tag{23}$$

This iterative process continues until the control sequence practically converges. Convergence can be tested by computing

$$w = \left(\sum_{\kappa=0}^{N-1} d_\kappa^2 \right)^{1/2} \tag{24}$$

and verifying that it is negligibly small.

Because of the nature of the Newton optimization direction and the Armijo stepsize selection rule, the previous algorithm is characterized by the following convergence properties:

1. Every limit control sequence $\{u_\kappa^*, \kappa = 0, 1, \cdots, N-1\}$ is stationary; namely, it satisfies the conditions of local optimality.

2. If the problem is convex, the convergence rate is faster than superlinear with order two; namely, the norm $\|u - u^*\|$, where u is the iteration and u^* is the optimal control sequence, converges to zero faster than all sequences of the form $r_i = q\beta^p$, where $q > 0$, $\beta \in (0, 1)$, and $p \in (1, 2)$.

3. In the vicinity of a local minimum, the Armijo test is satisfied with $m = 0$, and the stepsize α equals one, avoiding multiple performance index evaluations.

As will be seen in the case studies section, these theoretical properties translate practically into an efficient optimization procedure.

The above control model does not explicitly handle constraints (3*a*) and (3*b*). Herein, these constraints will be handled implicitly by adding appropriate performance index terms which penalize the deviations of the state and control variables from certain desirable trajectories. The procedure is demonstrated in the following section.

5. APPLICATION

The previous control model is used to obtain real time optimal irrigation rates over the growing season. We use the parameters for the maize crop as reported by *de Wit et al.* [1978] and *Protopapas and Bras* [1988]. Extensive field experiments have been reported in the past for the calibration and verification of the soil-crop-climate simulation model for several crops at different sites.

5.1. *System Dynamics*

To derive (6) we need to expand the propagation equation (1) for each plant and soil state variable in Taylor's series around the nominal values. Keeping only the first order terms, we get

$$\begin{bmatrix} x_{1,\kappa+1} \\ x_{2,\kappa+1} \\ \vdots \\ x_{6,\kappa+1} \end{bmatrix} = \begin{bmatrix} f_{1,\kappa}(\hat{x}_\kappa, \hat{u}_\kappa) \\ f_{2,\kappa}(\hat{x}_\kappa, \hat{u}_\kappa) \\ \vdots \\ f_{6,\kappa}(\hat{x}_\kappa, \hat{u}_\kappa) \end{bmatrix} + \begin{bmatrix} \frac{\partial f_{1,\kappa}}{\partial x_{1,\kappa}} & \frac{\partial f_{1,\kappa}}{\partial x_{2,\kappa}} & \cdots & \frac{\partial f_{1,\kappa}}{\partial x_{6,\kappa}} \\ \frac{\partial f_{2,\kappa}}{\partial x_{1,\kappa}} & \frac{\partial f_{2,\kappa}}{\partial x_{2,\kappa}} & \cdots & \frac{\partial f_{2,\kappa}}{\partial x_{6,\kappa}} \\ & \vdots & & \\ \frac{\partial f_{6,\kappa}}{\partial x_{1,\kappa}} & \frac{\partial f_{6,\kappa}}{\partial x_{2,\kappa}} & \cdots & \frac{\partial f_{6,\kappa}}{\partial x_{6,\kappa}} \end{bmatrix}$$

$$\cdot \begin{bmatrix} x_{1,\kappa} - \hat{x}_{1,\kappa} \\ x_{2,\kappa} - \hat{x}_{2,\kappa} \\ \vdots \\ x_{6,\kappa} - \hat{x}_{6,\kappa} \end{bmatrix} + \begin{bmatrix} \frac{\partial f_{1,\kappa}}{\partial u_{1,\kappa}} \\ \frac{\partial f_{2,\kappa}}{\partial u_{1,\kappa}} \\ \vdots \\ \frac{\partial f_{6,\kappa}}{\partial u_{1,\kappa}} \end{bmatrix} (u_\kappa - \hat{u}_\kappa)$$

or

$$\delta x_{\kappa+1} = A_\kappa \delta x_\kappa + B_\kappa \delta u_\kappa$$

The evaluation of matrix A_κ and vector B_κ requires the differentiation of the state functions with respect to the state and control variables. Some state propagation equations are complicated functions, making this necessary step laborious and algebraically demanding. The derivation is given by *Protopapas* [1988, Appendix D]. Two problems associated with linearization are (1) the derivatives of some functions used in the plant model are stepwise discontinuous functions, and (2) the propagation equation for the plant water potential changes under different conditions, depending on whether the photosynthesis or the transpiration process limits plant growth. We have checked the linearization by comparing the nonlinear simulations to the predictions of the linear dynamics, and the results are satisfactory. Given the small simulation time step and the smooth state trajectories in our system, the above two possible problems do not have significant impact.

5.2. *Performance Index*

The objective function of the irrigation scheduling problem should reflect the trade-off between a desirable optimum crop yield and the use of minimum cost for irrigation. Three alternative quadratic performance indices which meet these requirement are presented next:

5.2.1. *Final state target and minimum irrigation cost at each stage.*

$$J = \tfrac{1}{2}(x_N - \bar{x}_N)^T \bar{Q}_N (x_N - \bar{x}_N) + \sum_{\kappa=0}^{N-1} \tfrac{1}{2} \bar{R}_\kappa (u_\kappa - \bar{u}_\kappa)^2 \tag{25}$$

where $\bar{x}_N$ is the target state corresponding, for example, to the record yield year, and $\bar{u}_\kappa$ is the sequence of no-cost irrigation policy. The quadratic Taylor's series expansion is

$$\tilde{J} = \tfrac{1}{2}\,\delta x_N^T Q_N \delta x_N + (\hat{x}_N - \bar{x}_N)^T Q_N \delta x_N + \sum_{\kappa=0}^{N-1} \tfrac{1}{2} R_\kappa \delta u_\kappa^2 + (\hat{u}_\kappa - \bar{u}_\kappa) R_\kappa \delta u_\kappa$$

so that

$$Q_N = \bar{Q}_N \qquad a_N = \bar{Q}_N(\hat{x}_N - \bar{x}_N) \qquad Q_\kappa = 0$$

$$a_\kappa = 0 \qquad M_\kappa = 0$$

$$R_\kappa = \bar{R}_\kappa \qquad b_\kappa = \bar{R}_\kappa(\hat{u}_\kappa - \bar{u}_\kappa)$$

5.2.2. *Tracking a target state and minimum irrigation cost at each stage.*

$$J = \tfrac{1}{2}\,(x_N - \bar{x}_N)^T \bar{Q}_N (x_N - \bar{x}_N) + \sum_{\kappa=0}^{N-1} \tfrac{1}{2} \bar{R}_\kappa (u_\kappa - \bar{u}_\kappa)^2 + \tfrac{1}{2}\,(x_\kappa - \bar{x}_\kappa)^T \bar{Q}_\kappa (x_\kappa - \bar{x}_\kappa) \qquad (26)$$

with quadratic approximation

$$\tilde{J} = \tfrac{1}{2}\,\delta x_N^T Q_N \delta x_N + (\hat{x}_N - \bar{x}_N)^T Q_N \delta x_N + \sum_{\kappa=0}^{N-1} \tfrac{1}{2} R_\kappa \delta u_\kappa^2 + (\hat{u}_\kappa - \bar{u}_\kappa) R_\kappa \delta u_\kappa + \tfrac{1}{2}\,\delta x_\kappa^T Q_\kappa \delta x_\kappa + (\hat{x}_\kappa - \bar{x}_\kappa)^T Q_\kappa \delta x_\kappa$$

so that

$$Q_N = \bar{Q}_N \qquad a_N = \bar{Q}_N(\hat{x}_N - \bar{x}_N)$$

$$Q_\kappa = \bar{Q}_\kappa \qquad a_\kappa = \bar{Q}_\kappa(\hat{x}_\kappa - \bar{x}_\kappa) \qquad M_\kappa = 0$$

$$R_\kappa = \bar{R}_\kappa \qquad b_\kappa = \bar{R}_\kappa(\hat{u}_\kappa - \bar{u}_\kappa)$$

5.2.3. *Final state target and minimum production cost at each stage.*

$$J = \tfrac{1}{2}\,(x_N - \bar{x}_N)^T \bar{Q}_N (x_N - \bar{x}_N) + \sum_{\kappa=0}^{N-1} \tfrac{1}{2}\,(q_\kappa^T x_\kappa - r_\kappa u_\kappa)^2$$

with quadratic approximation

$$\tilde{J} = \tfrac{1}{2}\,\delta x_N^T Q_N \delta x_N + (\hat{x}_N - \bar{x}_N)^T Q_N \delta x_N + \sum_{\kappa=0}^{N-1} \tfrac{1}{2} r_\kappa^2 \delta u_\kappa^2 + \delta u_\kappa(-r_\kappa q_\kappa^T)\delta x_\kappa + \tfrac{1}{2}\,\delta x_\kappa^T (q_\kappa^T q_\kappa) \delta x_\kappa$$

so that

$$Q_N = \bar{Q}_N \qquad a_N = \bar{Q}_N(\hat{x}_N - \bar{x}_N)$$

$$Q_\kappa = q_\kappa^T q_\kappa \qquad a_\kappa = 0$$

$$R_\kappa = r_\kappa^2 \qquad b_\kappa = 0 \qquad M_\kappa = -r_\kappa q_\kappa^T$$

Depending on the relative magnitudes of $\{Q_\kappa, R_\kappa, q_\kappa$, and r_κ, $\kappa = 0, 1, \cdots, N\}$ the previous performance indices can place priority to the final yield, the irrigation cost, or the net gain. The associated trade-offs may also be derived through sensitivity analysis. In addition, the state and control cost terms can help generate sequences satisfying constraints (3*a*) and (3*b*).

5.3. *Results*

In the following we solve characteristic cases of the irrigation scheduling problem using the linear quadratic control approach. The parameters for the maize crop are as in the paper by *Protopapas and Bras* [1988]. The soil is a Panoche clay with hydraulic parameters given in the appendix. The initial water potential in the soil is −1000 cm, corresponding to an initial soil moisture of 0.12. The depth of the root zone is set equal to 50 cm in all cases. The horizon over which hourly irrigation rates are to be decided is 50 days or 1200 time steps. The problem involves six state variables and one control variable. The objective is to track the crop shoot weight, which is a result of constant irrigation at a rate of 0.5 cm/day and zero soil salinity. These conditions yield the biologically maximum achievable yield. The target state tracking objective is limited by the goal of keeping the irrigation rate as close to zero as possible. This trade-off is implemented by using the second objective function of the previous section. The coefficients Q_κ and R_κ reflect the relative weight attributed to the deviations of the state and the control from their target values. We use $Q_\kappa(3, 3) = 1$ (dollar per kilogram per hectare)^2, the other entries of matrix Q_κ being zero, and $R_\kappa = r^2(\Delta t/86{,}400)^2$ (dollars per centimeter of irrigation water)^2, with r^2 taking values from 10^2 ("low cost" irrigation water) to 10^8 ("high cost" irrigation water). The cost coefficients reflect the relative importance that the user attributes to the deviations of the state and control sequences from the desirable trajectories.

Case 1: Initial Salinity = 0.0; Irrigation Water Salinity = 0.0. This pilot run serves as a verification of the approach. For low cost control ($r^2 = 10^2$) the controller finds the sequence of irrigation rates that result in the target yield trajectory, that is, 0.5 cm/d (Figure 3*a*). The total water used is 48 cm, of which 30 cm are drained below the root zone, 8 cm are transpired, and 4 cm are evaporated from the soil surface, increasing the water stored in the root zone by 6 cm (Figure 3*b*).

For expensive control ($r^2 = 10^8$) the controller suggests initial irrigation at a rate of 0.5 cm/d, which normally results in the desirable yield. Subsequently, it gradually forces the rate to values close to zero (Figure 4*a*). The use of water in this case is restricted because of high irrigation costs. The application rates reflect the necessity of keeping the state on track initially so that its inevitable deviations from the target values at later stages are minimized. The water use is 16 cm, percolation is 2 cm, evapotranspiration is 12 cm, and the change in storage is 2 cm (Figure 4*b*).

As expected at the limit for $r^2 \rightarrow 0$, the control sequence is constant at 0.5 cm/d, while for $r^2 \rightarrow \infty$ the optimal decision is "no irrigation." For this run the effects of the initial nominal control sequence and of the stepsize selection rule for the resulting optimal rates were also studied. In both cases the method is found accurate and robust, although the number of iterations required to reach the solution varies. In this study a mainframe CDC-Cyber computer was used, and the execution time for the case at hand varied from 2 to 4 min.

Case 2: Initial Salinity = 0.0; Irrigation Water Salinity = 1000.0 mg/L. For $r^2 = 10^4$, as shown in Figure 5*a*, the

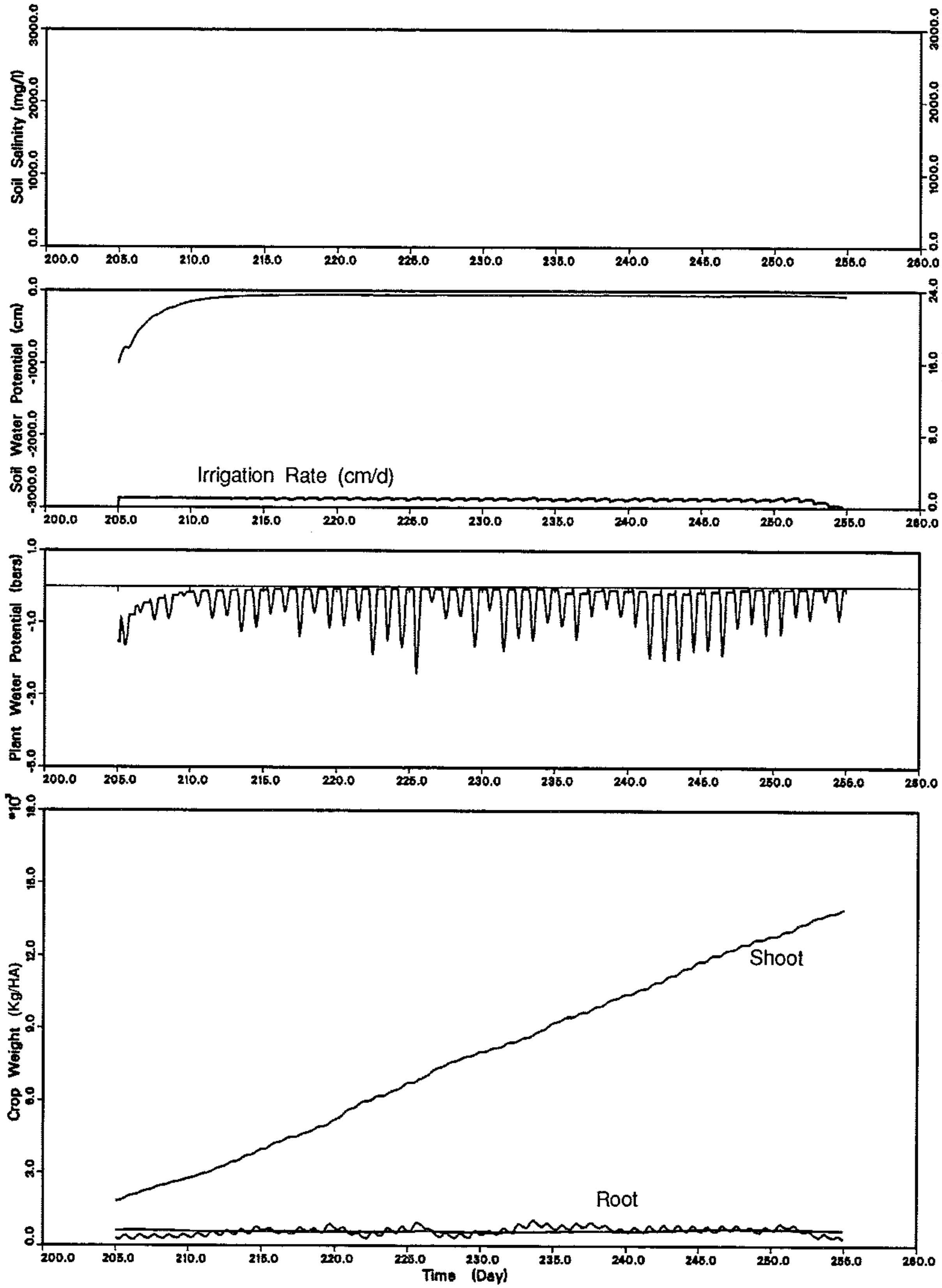

Fig. 3*a*. State and control sequences (initial salinity = 0.0; irrigation water salinity = 0.0; low cost irrigation).

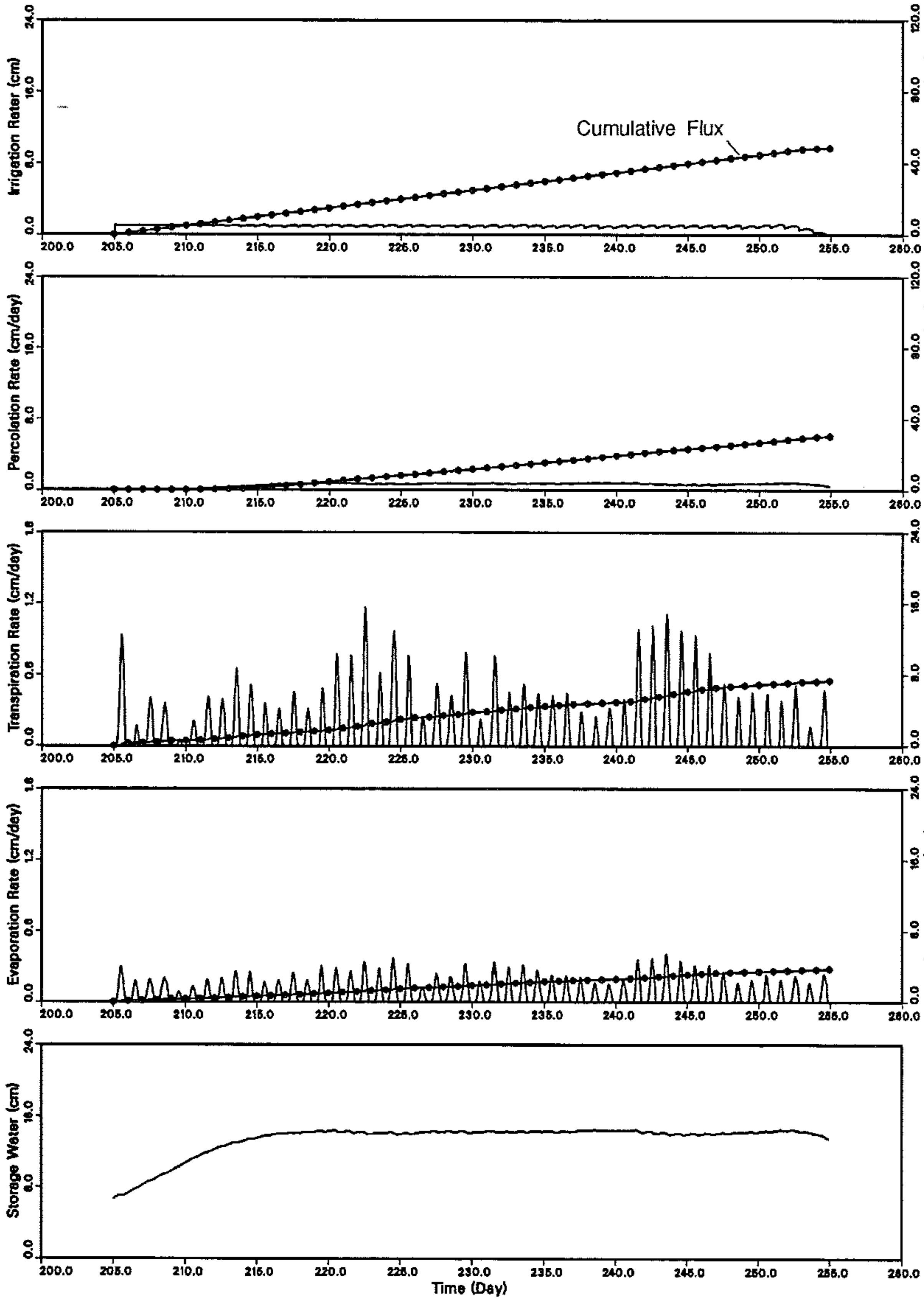

Fig. 3*b*. Hydrologic fluxes and water storage (initial salinity = 0.0; irrigation water salinity = 0.0; low cost irrigation).

controller initially suggests a high irrigation rate, which changes the matric potential in the soil from −1000 to −200 cm and quickly increases the water storage by 4 cm. Since the irrigation water is saline, this action also results in rapid increase of the soil salinity. Yet, the effective soil potential, which includes the osmotic effects of salinity, is −350 cm (much less negative than its initial value). In the following days the optimal sequence simply covers the percolation and evapotranspiration water losses by applying frequent low rate irrigations, which keep the water stored in the root zone almost constant. After every irrigation the soil salinity increases, and by day 230 it equals the salinity of the irrigation water. In the following period until the end of the season the optimal rates reflect the goals of stabilizing the salinity level at 1000 mg/L, the soil potential at −650 cm, and the water storage at 20 cm. This "optimal steady state" condition corresponds to minimum objective function, balancing the deviations of the state and the control. The total irrigation water exceeds 120 cm, most of which is lost below the root zone (Figure 5*b*). The solution required 61 iterations and 35 min CPU time.

For $r^2 = 10^2$ and 10^6 the pattern is similar, although the irrigation rates vary accordingly. For $r^2 = 10^8$ the contribution of the control outweighs the state cost, and after irrigating for 2 days at about 2 cm/d initially, the required rate is about 0.5 cm/d for the remaining period. This is sufficient to maintain the water potential at a small negative value (wet soil conditions), while salts are inevitably left to gradually accumulate after each irrigation.

It is well documented that the absorbing capability of the roots decreases in very dry and also in very wet conditions. The physical model accounts for overirrigation by assuming that the water uptake by the roots decreases if the effective soil potential falls outside an optimum range between −300 and −50 cm (factor f_{ψ_s} in (A1)). The model predicts high irrigation rates for case 2 and for the following cases 4 and 5 (Figures 5, 7, and 8) where controlling the soil salinity is the critical objective. In these cases the effective soil potential stabilizes at a more negative value than −300 cm because the sensitivity of crop yield to high salinity levels is more important than the decrease in water uptake. The high irrigation rates are necessary to keep soil salinity at the lowest possible level, which is equal to the salinity of irrigation water (1000 mg/L).

Case 3: Initial Salinity = 3000.0 mg/L; Irrigation Water Salinity = 0.0. For $r^2 = 10^4$, as shown in Figure 6*a*, the controller senses the possibility of using the salt-free water to rapidly leach the salts below the root zone. This is done in the first 20 days by a series of irrigation rates, that decrease in magnitude. Water storage increases by 9 cm at day 225. The effect of this action is to essentially keep the crop yield at its desirable value after the leaching of salts, and consequently, after day 225 the problem is similar to case 1. Indeed, for the rest of the season the optimal rate is set to about 0.5 cm/d. The total water used is 117 cm, the percolation loss is 100 cm, the evapotranspiration is 12 cm, and the change in storage is 5 cm (Figure 6*b*). The solution converged in eight iterations and 3 min CPU time. For $r^2 = 10^2$ the leaching is completed in 15 days, using higher irrigation rates, while for $r^2 = 10^6$ a total of 30 days is required.

In current irrigation practice, leaching of salts is typically done prior to the growing season and then a constant irrigation rate is applied during the season. The simulations in cases 3 and 5 (Figures 6 and 8) illustrate the ability of the method to quantitatively reproduce such practical irrigation rules.

Case 4: Initial Salinity = 1000.0 mg/L; Irrigation Water Salinity = 1000.0 mg/L. In this case, soil salinity cannot be reduced below its initial value of 1000 mg/L, since root zone water is lost, and the available irrigation water has the same high salinity. The best action is to irrigate so that the initial salinity does not increase. As shown in Figure 7*a* for $r^2 = 10^6$, a sequence of high irrigation rates succeed in stabilizing the soil salinity at 1000 mg/L throughout the season. More than 120 cm of irrigation water are applied, and the storage water is maintained at about 21 cm, with daily fluctuations following the periodic evapotranspiration loss (Figure 7*b*). The solution was found in 12 iterations and 4 min CPU time. For less expensive water the optimal sequences also reflect the goal to keep the soil salinity close to its initial value by continuous irrigation rates even higher than those shown in Figure 7*a*.

Case 5: Initial Salinity = 3000.0 mg/L; Irrigation Water Salinity = 1000.0 mg/L. This case demonstrates features similar to those of cases 3 and 4. The optimal irrigation sequence ($r^2 = 10^6$) is such that within the first few days, high irrigation rates reduce the salinity to the value of the irrigation water, that is, 1000 mg/L. During the rest of the growing season a continuous periodic rate is applied to maintain the desirable steady state (Figures 8*a* and 8*b*).

6. CONCLUSIONS AND FUTURE DIRECTIONS

This paper is motivated by the need for efficient use of water in agriculture and presents a modern optimal control approach for the problem of real time irrigation scheduling. We use a physically based representation of the dynamics of the soil-crop-atmosphere system. The variables characterizing the crop and soil status are concurrently simulated with an integrated state space model. Soil moisture and salinity conditions, which synergistically control the plant water uptake, are obtained by using lumped parameter mass balance models for the root zone. Crop yield is predicted by explicitly modeling the plant growth processes, such as assimilation, respiration, and transpiration, which are driven by the climatic inputs. The control model is an analytical optimization method for multistage multidimensional sequential decision-making problems. It is suitable for systems with nonlinear dynamics and objective functions. The method is based on local iterative approximations of the nonlinear problem with a linear quadratic problem.

In this paper the formulation is restricted to the deterministic case, that is, uncertainty of the climatic inputs is not considered. A series of examples are presented, where optimal irrigation schedules are obtained on an hourly basis over the growing season. The examples are chosen to test the correct behavior of the controller under a variety of operating conditions. The results follow a priori expected patterns, which can probably be guessed by experience and knowledge of the system. In fact, irrigation managers may have to account empirically for more types of constraints and considerations than presented in this paper. However, this study demonstrates that the optimal policies can generally be derived quantitatively and systematically for any set of initial conditions and system parameters. This is a major contribution of this work, which can be the basis for modern automated irrigation systems.

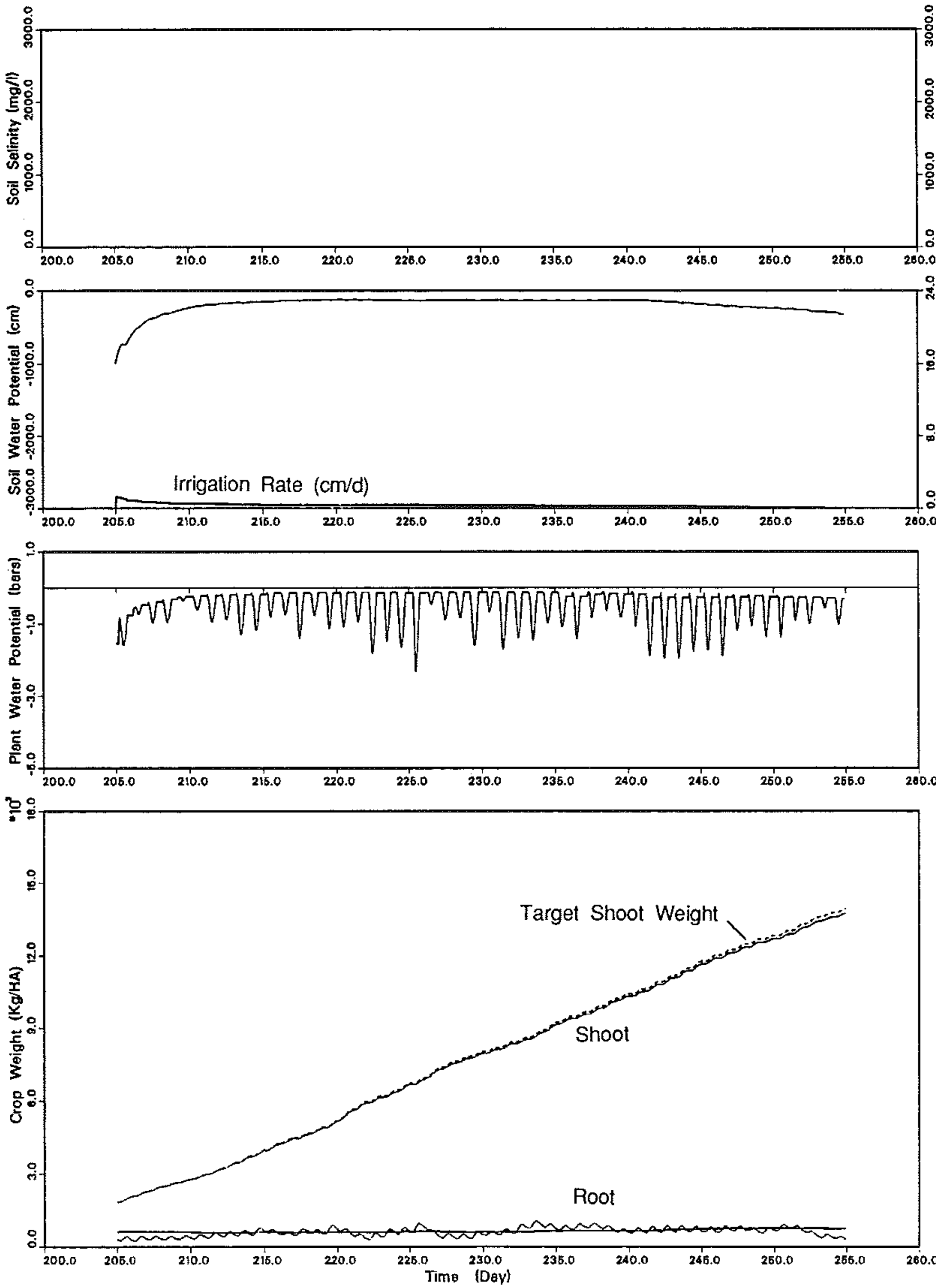

Fig. 4a. State and control sequences (initial salinity = 0.0; irrigation water salinity = 0.0; high cost irrigation).

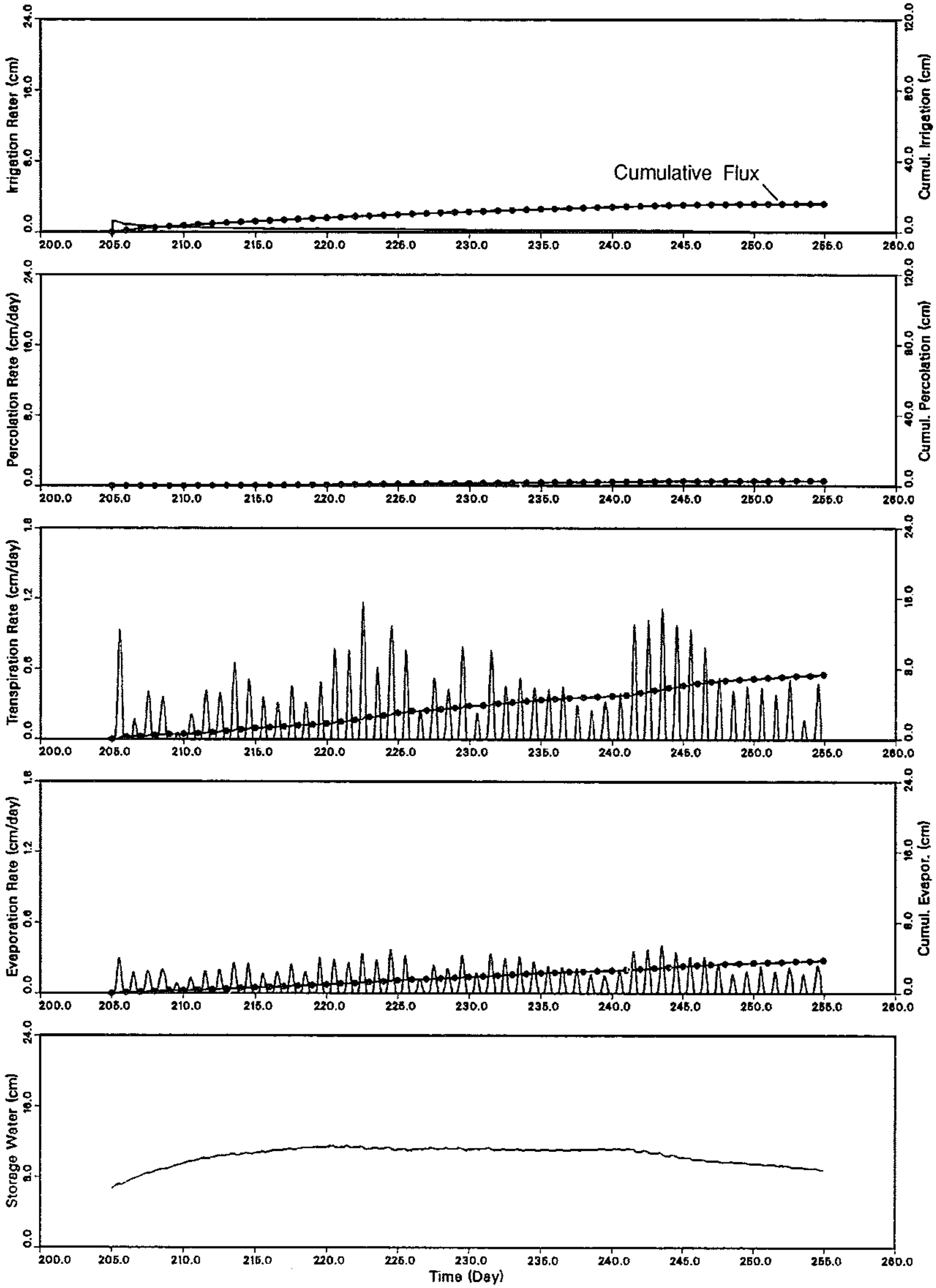

Fig. 4*b*. Hydrologic fluxes and water storage (initial salinity = 0.0; irrigation water salinity = 0.0; high cost irrigation).

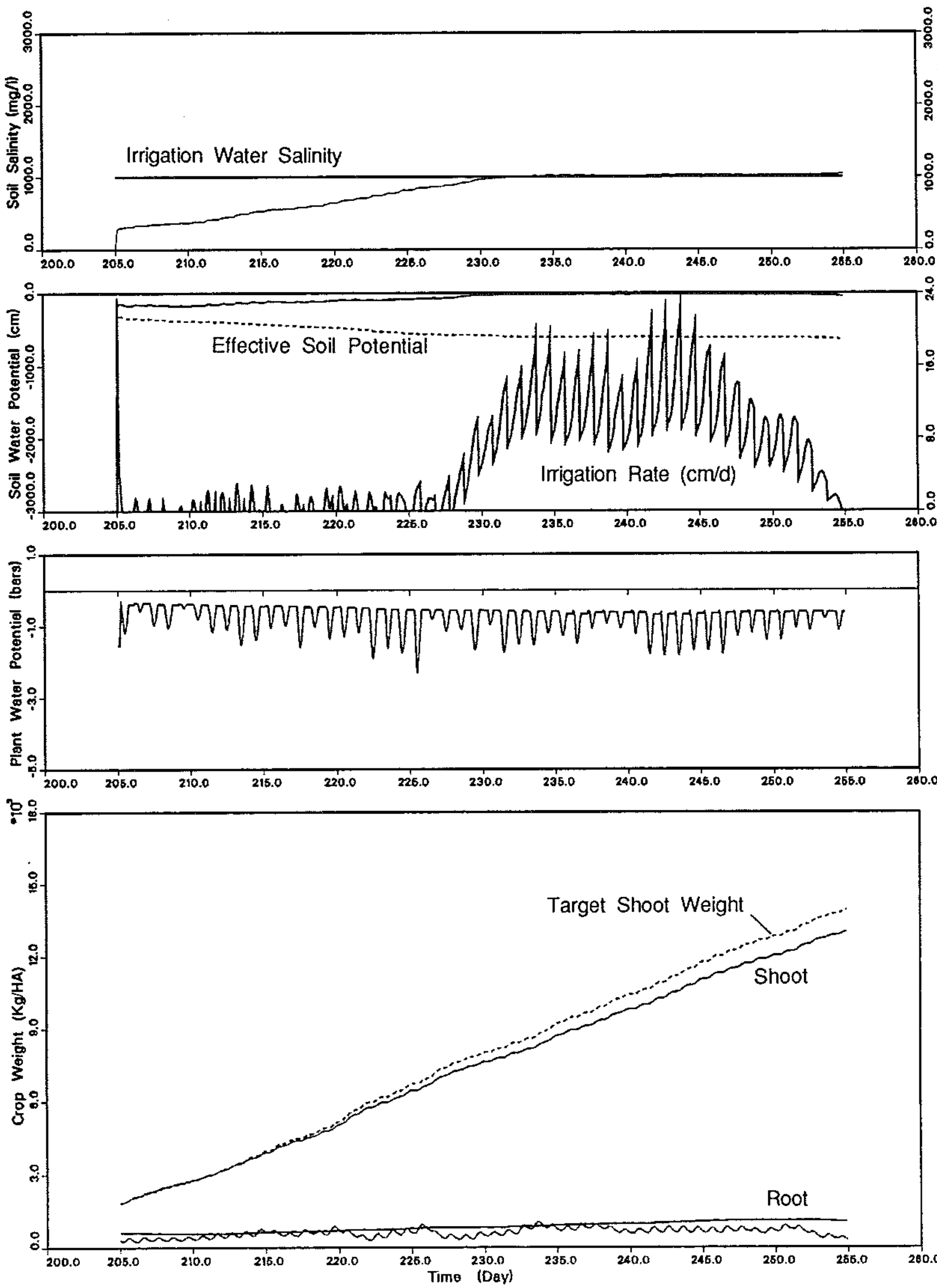

Fig. 5*a*. State and control sequences (initial salinity = 0.0; irrigation water salinity = 1000 mg/L).

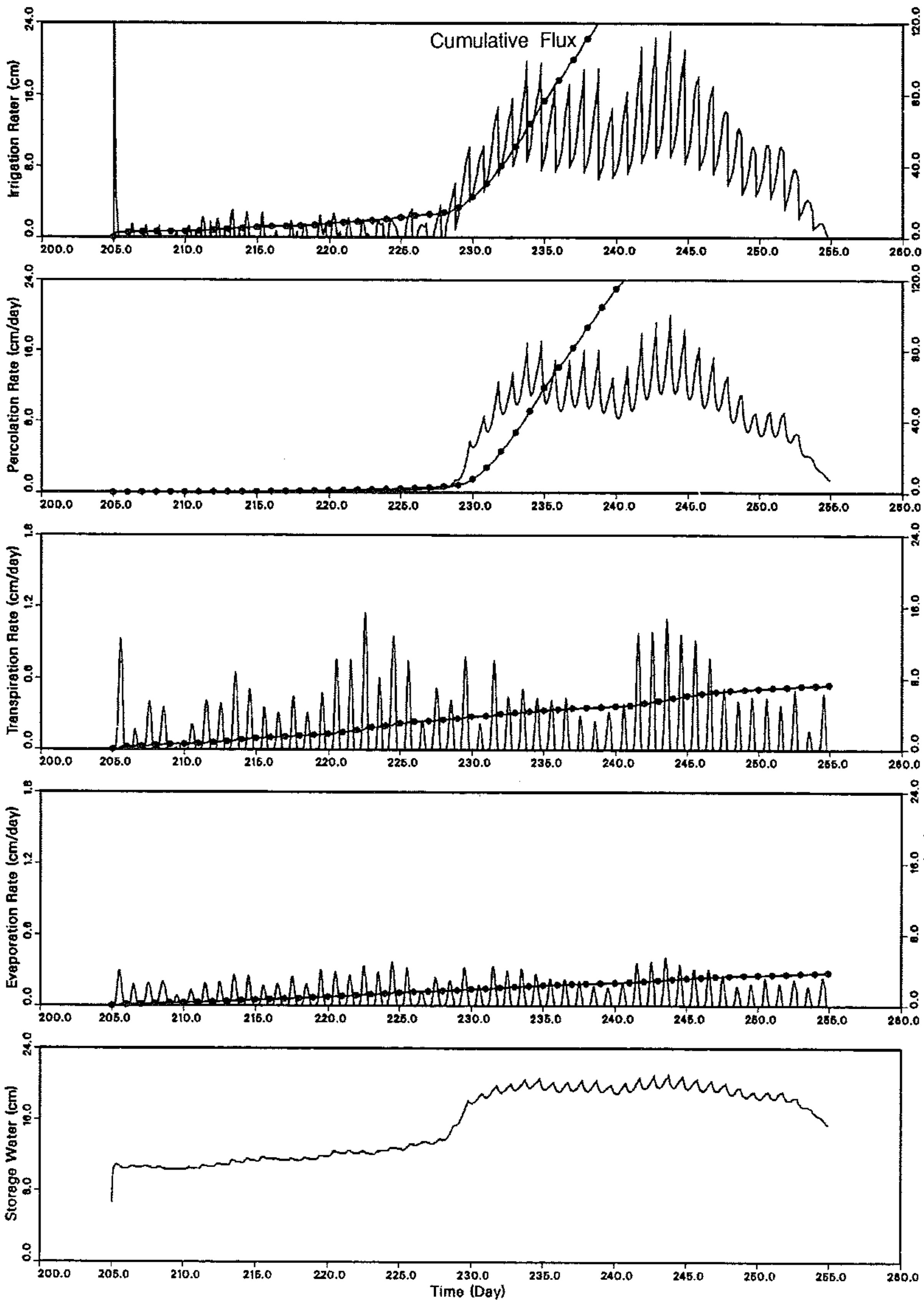

Fig. 5*b*. Hydrologic fluxes and water storage (initial salinity = 0.0; irrigation water salinity = 1000 mg/L).

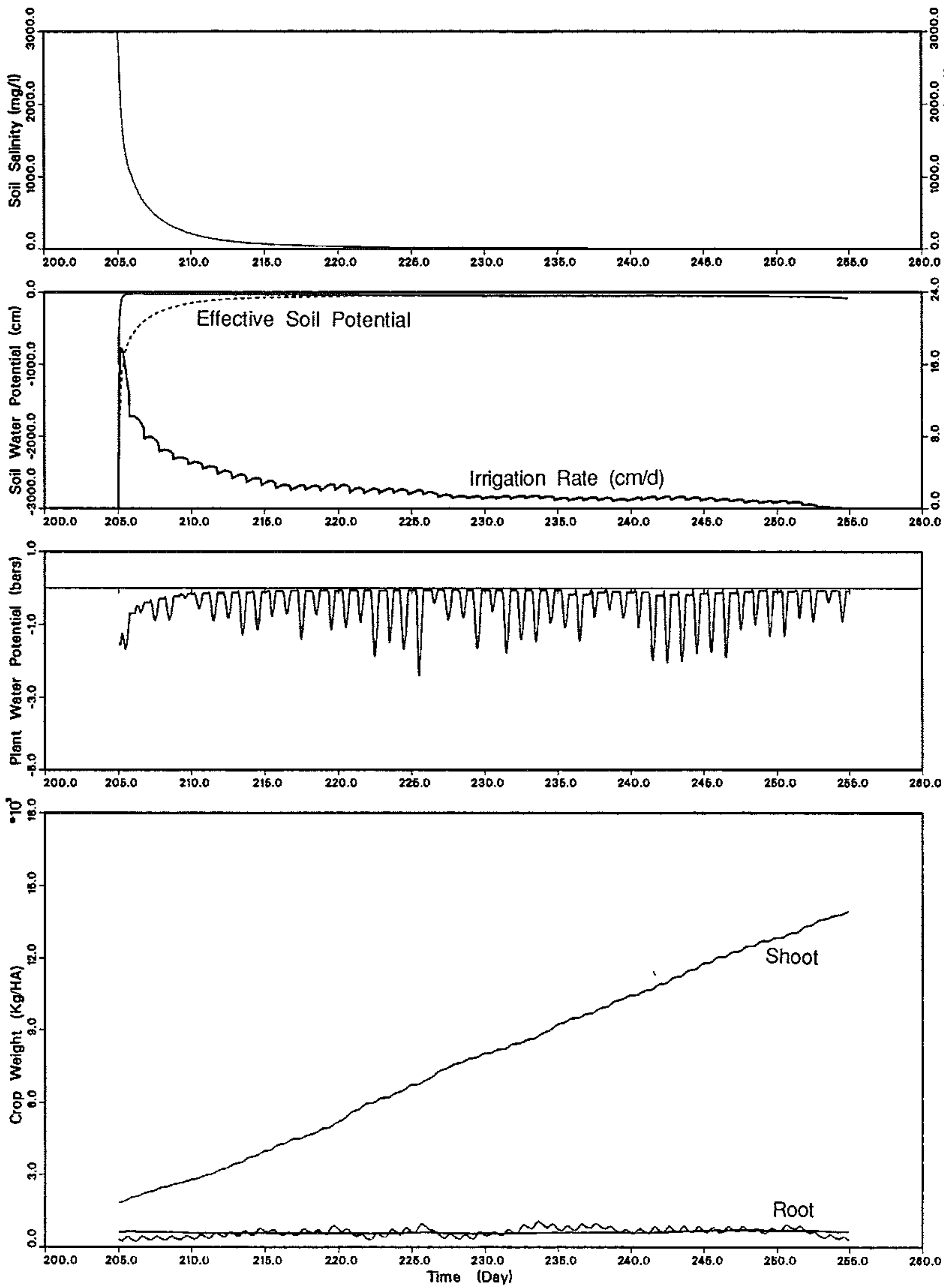

Fig. 6a. State and control sequences (initial salinity = 3000 mg/L; irrigation water salinity = 0.0).

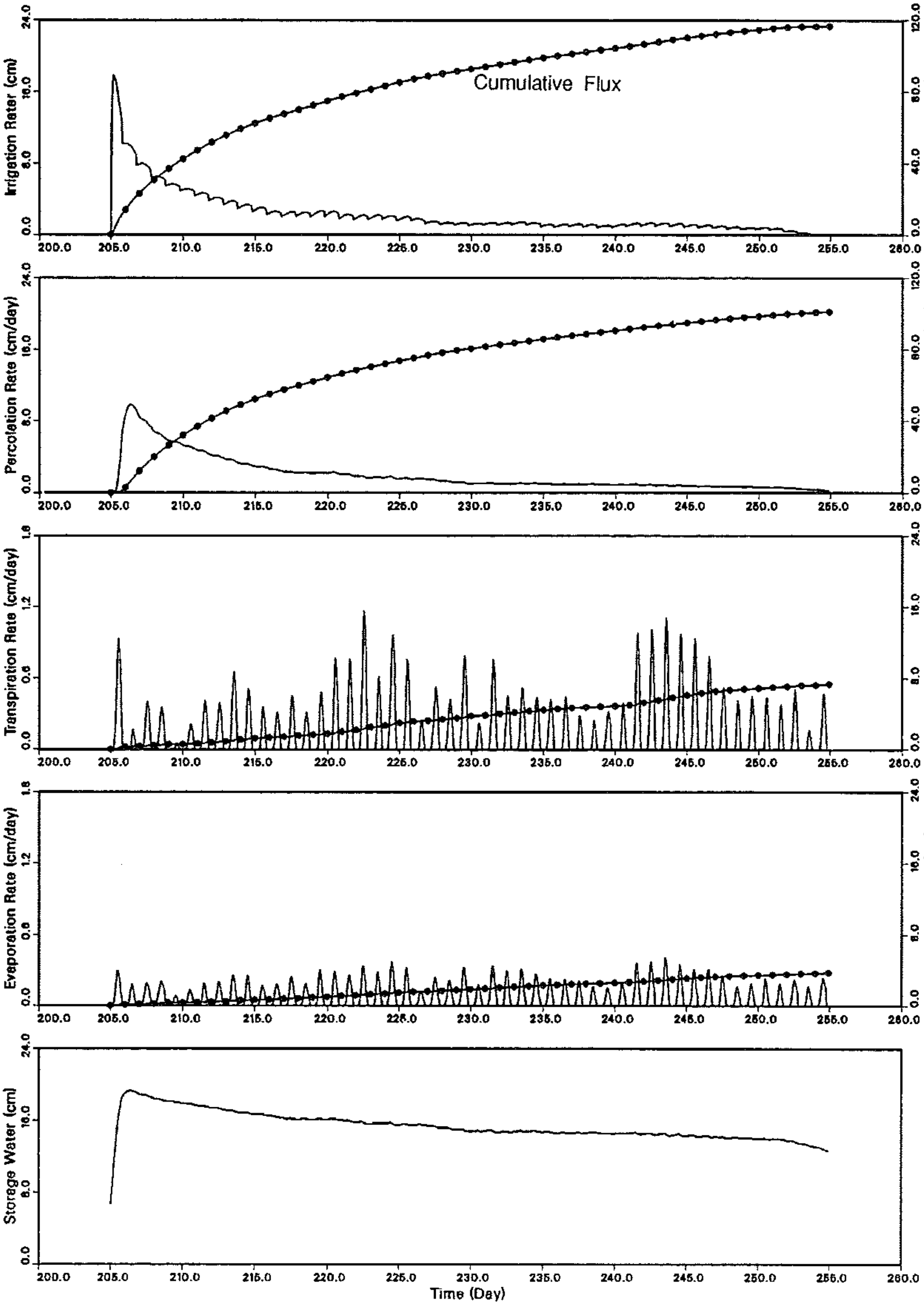

Fig. 6*b*. Hydrologic fluxes and water storage (initial salinity = 3000 mg/L; irrigation water salinity = 0.0).

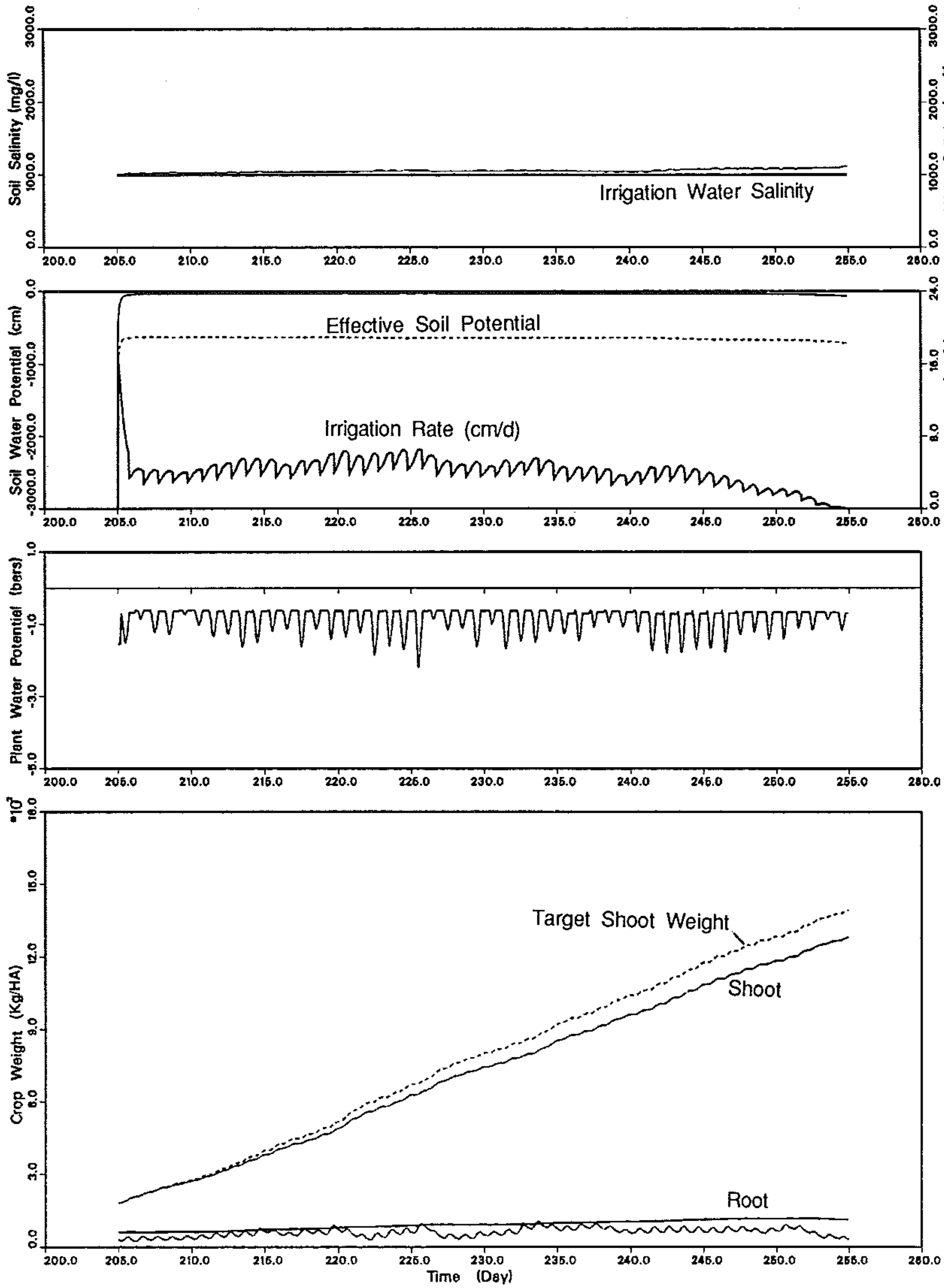

Fig. 7*a*. State and control sequences (initial salinity = 1000 mg/L; irrigation water salinity = 1000 mg/L).

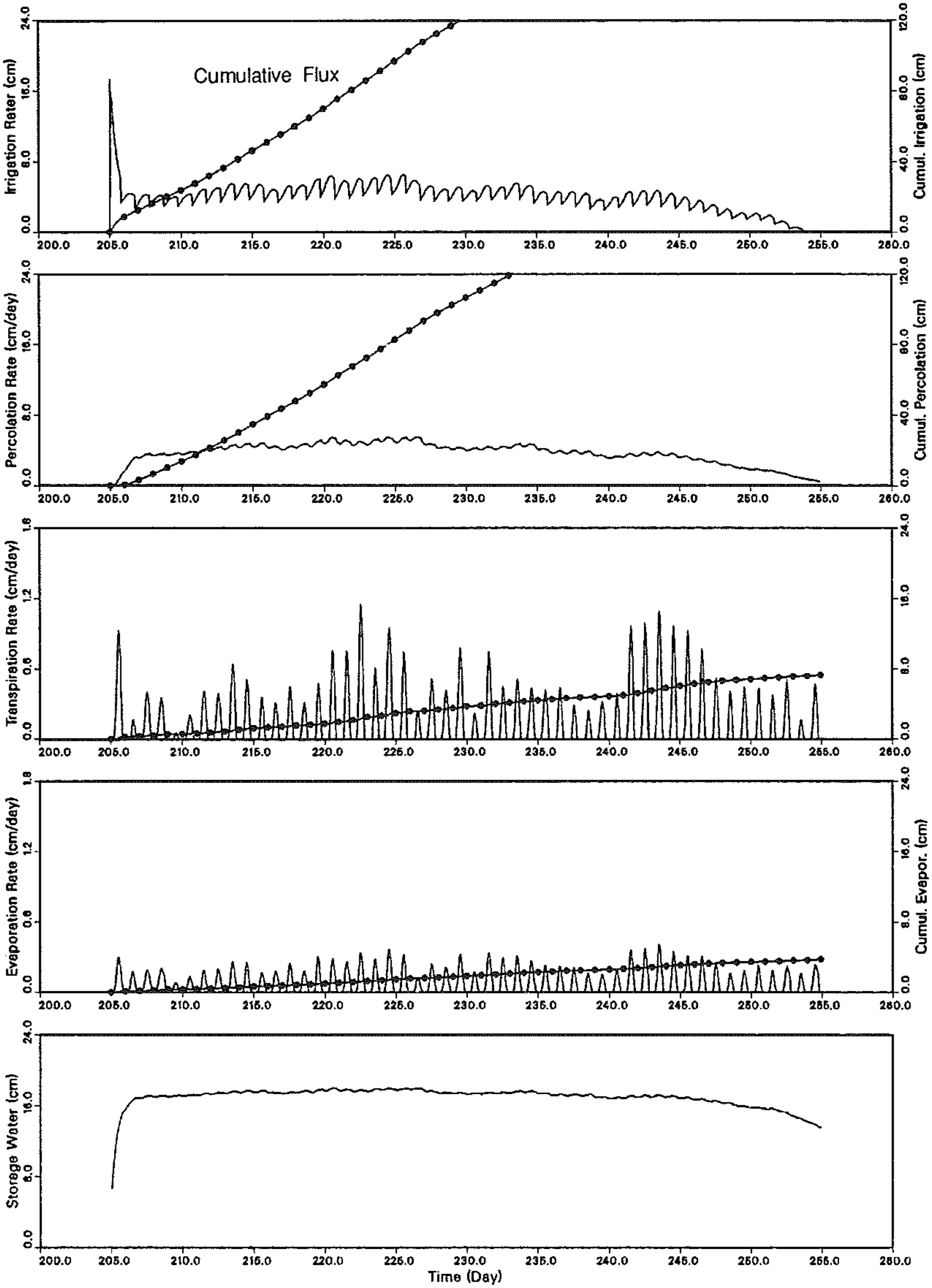

Fig. 7*b*. Hydrologic fluxes and water storage (initial salinity = 1000 mg/L; irrigation water salinity = 1000 mg/L).

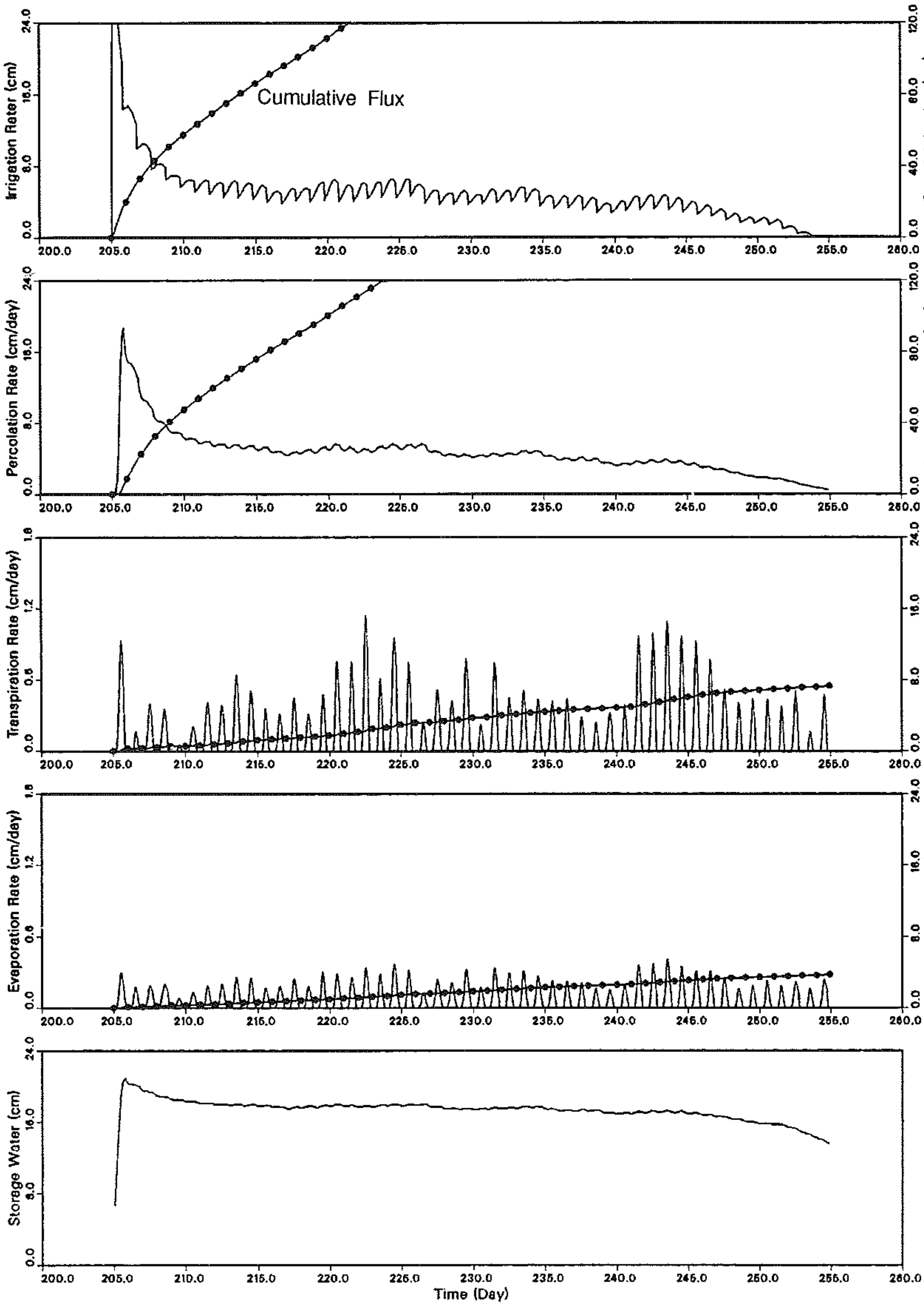

Fig. 8*a*. State and control sequences (initial salinity = 3000 mg/L; irrigation water salinity = 1000 mg/L).

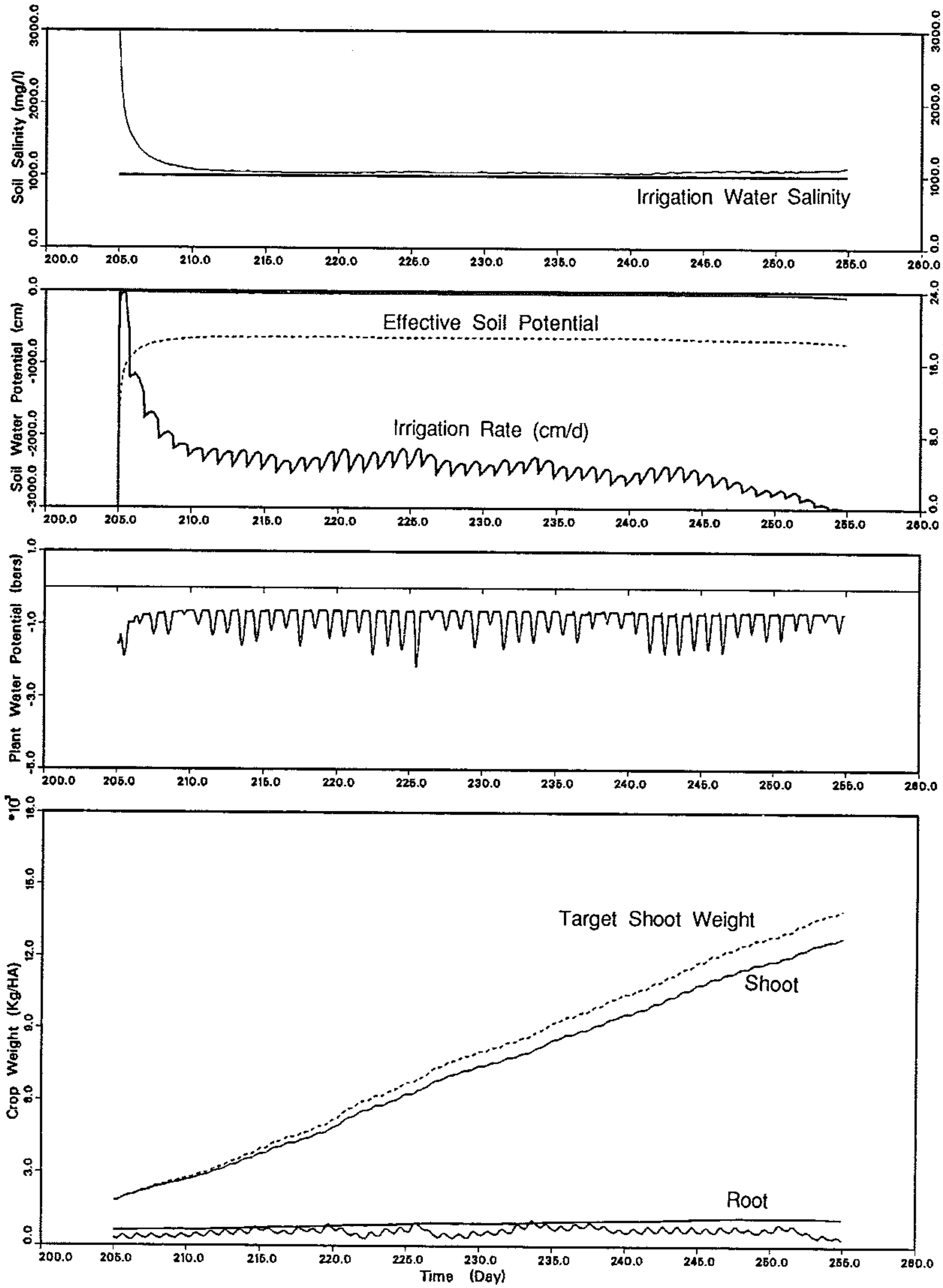

Fig. 8*b*. Hydrologic fluxes and water storage (initial salinity = 3000 mg/L; irrigation water salinity = 1000 mg/L).

An advantage of the methodology used is that each iteration generates a new control sequence, which is used for simulation with the nonlinear model and is accepted only if it improves the objective function. It is a feature of the Armijo stepsize selection rule that the objective function is always less or equal to its previous value after each iteration. Therefore no further simulations are necessary for checking the final sequence for optimality. The resulted control laws in the case studies are more detailed than any other laws that can be proposed by semiempirical or other methods. In some cases the results were favorably compared to different sensible irrigation policies.

However, it should be mentioned that the optimization model cannot account for inaccurate representation of the system dynamics. Although the proposed model is based on well-documented and validated physical principles, it is a conceptualization of the real system and is subjected to revision as more information becomes available. A powerful idea for remediating the system errors is the use of state estimation methods, which, however, require systematic observations of the state.

Finally, a variety of important future extensions are possible. Within the deterministic framework the available irrigation water may be fixed (resource allocation problem), the state variables may be bounded within extreme values (for example, maximum desirable salinity at the end of the season), the irrigation rate may be constrained by the existing on-farm technology, and different objective functions may be introduced. Such constraints can be incorporated in the control scheme. Similar considerations may also be examined within a stochastic framework, where the climatic inputs are random. Such problems can be successfully investigated using recently developed stochastic optimal control methods [*Georgakakos and Marks*, 1987; *Georgakakos*, 1989], and simple irrigation rules can be derived for implementation in irrigation practice.

APPENDIX: STATE SPACE FORMULATION OF THE SOIL-CROP-CLIMATE MODEL WITH SIMPLIFIED FLOW AND TRANSPORT DYNAMICS

In this appendix the propagation equations for the state variables of the soil-crop-climate model are summarized. The definition of the variables is summarized in the notation section following the appendix. For methodological purposes the plant and soil components of the model are presented separately.

A.1. *Plant State Variables*

The reader can find all the functions and parameter values that are not explicitly defined in the following in the paper by *Protopapas and Bras* [1988]. The above reference also provides the detailed conceptual basis for understanding the simplified description of this section. For some variables the notation of the associated computer code is used. During daytime, two cases are considered, depending on how the leaf resistance is computed.

A.1.1. *Equations during day period, case 1: photosynthesis controls transpiration.*

Plant water potential:

$$\psi_p(\kappa+1) = \frac{\text{WUPS} - E(\kappa)}{\text{ACRS}} \tag{A1}$$

where

$$\text{WUPS} = \psi_s(\kappa)\,\frac{W_R}{K_R}\,f_T(T_s)f_{\psi_s}(\psi_s)$$

$$\text{ACRS} = \frac{W_R}{K_R}\,f_T(T_s)f_{\psi_s}(\psi_s)$$

$$E(\kappa) = \frac{\{\Delta(R_n/\text{LAI}) + [e_s(T_a) - e_s(T_d)](\rho C_p/r_b)\}\text{LAI}/L}{\Delta + \gamma[1 + (r_l/r_b)]}$$

$$r_l(\kappa) = \frac{68.4\ (c_e - c_i)}{1.66\ \text{NCRIL}} - \frac{1.32}{1.66}\,r_b \qquad r_b = \frac{\alpha}{(u)^{1/2}}$$

$$\text{NCRIL} = (F_m - F_d)\left[1 - \exp\left(-\frac{\varepsilon R_v(\kappa)}{\text{LAI}\ F_m}\right)\right] + F_d$$

$$= F_n(\kappa)\,\frac{3600.}{\text{LAI}\ (\kappa)}$$

$$F_m(\kappa) = F_m^* f_r(\text{RL})$$

$$\text{RL}(\kappa) = \frac{\text{RES}}{\text{RES} + W_S + W_R}$$

Reserve weight:

$$\begin{aligned}\text{RES}\ (\kappa+1) = {} & \text{RES}\ (\kappa)\{1 - r_c c_G \Delta t g_T[T_c(\kappa)]g_p[\psi_p(\kappa+1)] \\ & - r_c c_G \Delta t g_T(T_s(\kappa))[1 - g_p(\psi_p(\kappa+1))]\} \\ & + \frac{F_n(\kappa)}{1.629}\,\Delta t - c_M \Delta t\{W_S(\kappa)m[T_c(\kappa)]\} \\ & + W_R(\kappa)m[T_s(\kappa)]\end{aligned} \tag{A2}$$

where $E(\kappa)$, $r_l(\kappa)$, NCRIL(κ), $F_m(\kappa)$, and RL(κ) are as before, and

$$F_n(\kappa) = \frac{\text{NCRIL}}{3600.}\,\text{LAI}$$

$$T_c(\kappa) = T_a(\kappa) + [R_n(\kappa) - \text{LE}\ (\kappa)]\ r_b(\kappa)/\text{LAI}\rho\text{C}_\text{p}$$

Shoot weight:

$$W_S(\kappa+1) = W_S(\kappa) + \text{RES}\ (\kappa) r_c g_T[T_c(\kappa)]g_p[\psi_p(\kappa+1)]\Delta t \tag{A3}$$

where $T_c(\kappa)$, $E(\kappa)$, $r_l(\kappa)$, NCRIL(κ), $F_m(\kappa)$, and RL(κ) are as before.

Root weight:

$$\begin{aligned}W_R(\kappa+1) = {} & \left\{1 - \frac{\Delta t}{\tau_s}\,f_T[T_s(\kappa)]\right\}W_R(\kappa) \\ & + \text{RES}(\kappa) r_c \Delta t g_T[T_s(\kappa)]\{1 - g_p[\psi_p(\kappa+1)]\}\end{aligned} \tag{A4}$$

A.1.2. *Equations during day period, case 2: transpiration controls photosynthesis.*

Plant water potential:

$$\psi_p(\kappa+1) = \frac{-\text{EEE2} + \text{EEE4}}{2\ \text{EEE1}} \tag{A5}$$

where

$$\text{EEE2} = \varepsilon_2(\varepsilon_4\alpha_2 + \varepsilon_5) + (-\varepsilon_4\varepsilon_1 + \varepsilon_3)\alpha_1$$

$$\text{EEE4} = (\text{EEE2}^2 - 4\ \text{EEE1 EEE3})^{1/2}$$

$$\text{EEE1} = \varepsilon_2\varepsilon_4\alpha_1$$

$$\text{EEE3} = \varepsilon_3\alpha_2 - \varepsilon_1(\varepsilon_4\alpha_2 + \varepsilon_5)$$

$$\varepsilon_1 = \text{WUPS} = \psi_s(\kappa)(W_R/K_R)f_T(T_s)f_{\psi_s}(\psi_s)$$

$$\varepsilon_2 = \text{ACRS} = (W_R/K_R)f_T(T_s)f_{\psi_s}(\psi_s)$$

$$\varepsilon_3 = \frac{\text{LAI}}{L}\left\{\Delta(\kappa)\frac{R_n}{\text{LAI}} + [e_s(T_a) - e_s(T_d)]\frac{\rho C_p}{r_b}\right\}$$

$$\varepsilon_4 = \Delta(\kappa) + \gamma$$

$$\varepsilon_5 = \gamma/r_b$$

Reserve weight: The reserve weight is given, as in case I, by (A1), but now the leaf resistance introduces dependence of the canopy temperature on $\psi_p(\kappa + 1)$. Variables $T_c(\kappa)$ and $E(\kappa)$ are as before but

$$r_l(\kappa) = (\alpha_1\psi_p(\kappa + 1) + \alpha_2)^{-1}$$

$$\text{NCRIL}(\kappa) = \frac{68.4\ (c_e - c_i)}{1.66\ r_l + 1.32\ r_b} = F_n(\kappa)\frac{3600.}{\text{LAI}(\kappa)}$$

Shoot weight: The shoot weight is given, as in case I, (A3), but again the leaf resistance introduces dependence of the canopy temperature on $\psi_p(\kappa + 1)$.

Root weight: The root weight is given again by (A4).

Equations during night period.

Plant water potential:

$$\psi_p(\kappa + 1) = \psi_p(\kappa) \tag{A6}$$

Reserve weight: The reserve weight is given as previously by (A2), but now the canopy temperature does not depend on the transpiration rate and therefore on the leaf resistance. The assimilation does not depend on the state or on the climatic inputs. That is

$$T_c(\kappa) = T_a(\kappa) + \frac{R_n(\kappa)}{\text{LAI}}\frac{r_b(\kappa)}{\rho C_p}$$

$$F_n(\kappa) = -\frac{F_d}{3600.}\text{LAI}(\kappa)$$

Shoot weight: The shoot weight is again given by (A3) with the same remarks about canopy temperature that were made for the reserve weight.

Root weight: As in previous cases.

A.2. *Soil State Variables*

The lumped flow and transport models used in this paper are simplified versions of the distributed models in the previously mentioned reference. The lumped models are discussed in more detail in the following.

The conservation of mass of water and salts in a soil column of unit area and depth equal to the depth of the root zone is written as

$$z_w d\theta/dt = -E - e - p + i \tag{A7}$$

$$z_w d(c\theta)/dt = c_0 i - pc \tag{A8}$$

where z_w is the depth of the root zone (in centimeters); E is the transpiration rate, e is the evaporation rate, p is the percolation rate, and i is the irrigation rate, all in centimeters per day (E and e from the plant model need to be converted properly); θ is the soil water content; c is the soil salinity, and c_0 is the salinity of the irrigation water (in milligrams per liter).

The unsaturated flow in the soil is described by the relationships of soil water content and conductivity to the soil matric potential. The parameterization of *Dagan and Bresler* [1983] is used in the form

$$\theta(\psi) = \theta_r + (\theta_s - \theta_r)(\psi_w/\psi)^\beta$$

$$K(\psi) = K_s(\psi_w/\psi)^\eta$$

where θ_r is the residual water content, θ_s is the porosity, ψ_w is the bubbling air pressure (in centimeters), K_s is the saturated hydraulic conductivity (in centimeters per day), and β and η are soil dependent parameters. For Panoche clay soil used in our study the parameters are $\theta_r = 0.05$; $\theta_s = 0.43$; $\psi_w = -15$ cm; $K_s = 24$ cm/day; $\beta = 0.36$; and $\eta = 2.59$.

Introducing further the differential soil moisture capacity $C(\psi)$ as

$$C(\psi) = \frac{d\theta}{d\psi} = -\frac{\beta}{\psi}(\theta_s - \theta_r)\left(\frac{\psi_w}{\psi}\right)^\beta$$

and assuming that the column is drained freely, so that $p = K(\psi)$, the discretization of (A7) gives the following propagation equation for the matric potential:

$$\psi(\kappa + 1) = \psi(\kappa) + \frac{\Delta t}{z_w C[\psi(\kappa)]} \cdot \{-E(\kappa) - e(\kappa) - K[\psi(\kappa)] + i(\kappa)\} \tag{A9}$$

where Δt is the time step of the simulation (in days).

Simple algebraic manipulation and use of (A7) in (A8) yields

$$z_w\theta dc/dt = c_0 i + (E + e - i)c \tag{A10}$$

The discretization of (A10) gives the following propagation equation for the salt concentration:

$$c(\kappa + 1) = c(\kappa) + \frac{\Delta t}{z_w\theta(\psi(\kappa))}[E(\kappa) + e(\kappa) - i(\kappa)]c(\kappa) + \frac{\Delta t}{z_w\theta[\psi(\kappa)]}c_0 i(\kappa) \tag{A11}$$

NOTATION

- ψ_p plant water potential, bars.
- RES weight of reserves, kg/Ha.
- W_S weight of shoot, kg/Ha.
- W_R weight of root, kg/Ha.
- ψ soil matric potential, cm.
- c soil salinity, mg/L.

ψ_s effective soil potential, cm.
R_n net absorbed solar radiation, J/m^2s.
R_v photosynthetically active radiation, J/m^2s.
T_a air temperature, °C.
T_d dew point temperature, °C.
T_s soil temperature, °C.
u wind speed, m/s.
i irrigation rate, cm/d.
K_R conductive ability of root system per unit weight, kg/Ha bar.
f_T effect of soil temperature on root conductance.
f_{ψ_s} effect of effective soil potential on root conductance.
E transpiration rate, gr/m^2s or cm/d.
Δ slope of curve of saturated vapor pressure versus temperature, mbars/°C.
LAI leaf area index.
e_s saturated vapor pressure, mbars.
ρC_p volumetric heat capacity of air, J/m^3°C.
r_b boundary layer resistance, s/m.
r_l leaf resistance, s/m.
L latent heat of vaporization, J/g.
γ psychometric constant, mbars/°C.
c_e external CO_2 concentration, cm^3 CO_2/m^3.
c_i internal CO_2 concentration, cm^3 CO_2/m^3.
F_m maximum assimilation rate, kg CO_2/Ha hr.
F_n assimilation rate, kg CO_2/Ha s.
F_d dark respiration, kg CO_2/Ha hr.
ε photosynthetic efficiency at light compensation, kg CO_2/J.
F_m^* potential CO_2 assimilation rate, kg CO_2/Ha hr.
f_r effect of reserve level on maximum assimilation rate.
RL reserve level or weight of reserves over total weight.
$r_c c_G$ conversion rate of reserves to biomass, day^{-1}.
g_T effect of temperature on growth.
g_p effect of plant water potential on growth.
c_M starch requirement for maintenance, kg reserves/kg biomass d.
m effect of temperature on maintenance.
T_c canopy temperature, °C.
τ_s suberization rate of roots, s.
Δt time step of simulation, s or day.
z_w depth of the root zone, cm.
e evaporation rate, cm/d.
p percolation rate, cm/d.
θ soil water content.
c_0 salinity of the irrigation water, mg/L.
θ_r residual water content.
θ_s porosity.
ψ_w bubbling air pressure, cm.
K_s saturated hydraulic conductivity, cm/d.
K hydraulic conductivity, cm/d.
$C(\psi)$ differential soil moisture capacity, cm^{-1}.

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