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Study of the asymptotic behavior of the solutions of three systems of difference equations of exponential form

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ABSTRACT

In this paper we study the boundedness, the persistence and the asymptotic behavior of the positive solutions of the following systems of two difference equations of exponential form:

$$x_{n+1} = \frac{\alpha + \beta e^{-y_n}}{\gamma + y_{n-1}}, \quad y_{n+1} = \frac{\delta + \epsilon e^{-x_n}}{\zeta + x_{n-1}},$$
$$x_{n+1} = \frac{\alpha + \beta e^{-y_n}}{\gamma + x_{n-1}}, \quad y_{n+1} = \frac{\delta + \epsilon e^{-x_n}}{\zeta + y_{n-1}},$$
$$x_{n+1} = \frac{\alpha + \beta e^{-x_n}}{\gamma + y_{n-1}}, \quad y_{n+1} = \frac{\delta + \epsilon e^{-y_n}}{\zeta + x_{n-1}},$$

where $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$ are positive constants and the initial values x_{-1}, x_0, y_{-1}, y_0 are positive constants.

1. Introduction

In [26] the authors studied the boundedness, the asymptotic behavior, the periodicity and the stability of the positive solutions of the difference equation

$$y_{n+1} = \frac{\alpha + \beta e^{-y_n}}{\gamma + y_{n-1}},$$

where α, β, γ are positive constants and the initial values x_{-1}, x_0 are positive numbers.

Motivated by the above paper, we will investigate the boundedness, the persistence and the asymptotic behavior of the positive solutions of the following systems of difference equations of exponential form:

$$x_{n+1} = \frac{\alpha + \beta e^{-y_n}}{\gamma + y_{n-1}}, \quad y_{n+1} = \frac{\delta + \epsilon e^{-x_n}}{\zeta + x_{n-1}}, \tag{1.1}$$
$$x_{n+1} = \frac{\alpha + \beta e^{-y_n}}{\gamma + x_{n-1}}, \quad y_{n+1} = \frac{\delta + \epsilon e^{-x_n}}{\zeta + y_{n-1}}, \tag{1.2}$$
$$x_{n+1} = \frac{\alpha + \beta e^{-x_n}}{\gamma + y_{n-1}}, \quad y_{n+1} = \frac{\delta + \epsilon e^{-y_n}}{\zeta + x_{n-1}} \tag{1.3}$$

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where α, β, γ, δ, ϵ, ζ are positive constants and the initial values x_{-1}, x_0, y_{-1}, y_0 are positive constants. Also for applications in Mathematical Biology or Population Dynamics, we consider β to be the growth or reproduction rate of species x_n and ϵ to be the growth or reproduction rate of species y_n.

Difference equations and systems of difference equations of exponential form can be found in the following papers: [23,25,26,29–31,33,39,43]. Moreover, as difference equations have many applications in applied sciences there are many papers and books concerning theory and applications of difference equations, (for partial review of the theory of difference equations, systems of difference equations and their applications see [1–43] and the references sited therein).

2. Global behavior of solutions of three general systems

In this section we consider the following three general systems of two difference equations

$$x_{n+1} = f(y_n, y_{n-1}), \quad y_{n+1} = g(x_n, x_{n-1}), \tag{2.1}$$
$$x_{n+1} = f(x_{n-1}, y_n), \quad y_{n+1} = g(x_n, y_{n-1}), \tag{2.2}$$

and

$$x_{n+1} = f(x_n, y_{n-1}), \quad y_{n+1} = g(x_{n-1}, y_n), \tag{2.3}$$

where f, g are continuous functions and the initial values x_{-1}, x_0, y_{-1}, y_0 are positive numbers.

Working in a similar fashion as in relative theorems included in [10,11,17,24] we can state the following theorem which is useful for the study of our systems Eqs. (1.1)–(1.3).

Theorem 2.1. *Let f, g, $f: R^+ \times R^+ \to R^+$, $g: R^+ \times R^+ \to R^+$ be continuous functions, $R^+ = (0, \infty)$. Then the following statements are true*

(i) *Let a_1, b_1, a_2, b_2 be positive numbers such that $a_1 < b_1$, $a_2 < b_2$ and*

$$f : [a_2, b_2] \times [a_2, b_2] \to [a_1, b_1], \quad g : [a_1, b_1] \times [a_1, b_1] \to [a_2, b_2].$$

Suppose that the function $f(u,v)$ is a decreasing function with respect to u (resp. v) for all v (resp. u) and $g(z,w)$ is a decreasing function with respect to z (resp. w) for every w (resp. z). Finally suppose that if m, M, r, R are real numbers such that if

$$M = f(r, r), \quad m = f(R, R), \quad R = g(m, m), \quad r = g(M, M)$$

then $m = M$ and $r = R$. Then the system of difference Eq. (2.1) has a unique positive equilibrium $(\bar{x}, \bar{y})$ and every positive solution of the system Eq. (2.1) which satisfies

$$x_{n_0} \in [a_1, b_1], \quad x_{n_0+1} \in [a_1, b_1], \quad y_{n_0} \in [a_2, b_2], \quad y_{n_0+1} \in [a_2, b_2], \quad n_0 \in \mathbb{N} \tag{2.4}$$

tends to the unique positive equilibrium of Eq. (2.1).

(ii) Let

$$f : [a_1, b_1] \times [a_2, b_2] \to [a_1, b_1], \quad g : [a_1, b_1] \times [a_2, b_2] \to [a_2, b_2].$$

Suppose that the functions $f(x,y)$ and $g(x,y)$ are decreasing functions with respect to x (resp. y) for all y (resp. x). Finally suppose that if m, M, r, R are real numbers such that if

$$M = f(m, r), \quad m = f(M, R), \quad R = g(m, r), \quad r = g(M, R) \tag{2.5}$$

then $m = M$ and $r = R$. Then the systems of difference Eqs. (2.2) and (2.3) have a unique positive equilibrium $(\bar{x}, \bar{y})$ and every positive solution of the system Eqs. (2.2) (resp. (2.3)) satisfying Eq. (2.4) tends to the unique positive equilibrium of Eqs. (2.2) (resp. (2.3)).

Proof. (i) System Eq. (2.1) is equivalent to the system of separated equations

$$x_{n+1} = f(g(x_{n-1}, x_{n-2}), g(x_{n-2}, x_{n-3})) = F(x_{n-1}, x_{n-2}, x_{n-3}),$$
$$y_{n+1} = g(f(y_{n-1}, y_{n-2}), f(y_{n-2}, y_{n-3})) = G(y_{n-1}, y_{n-2}, y_{n-3}), \quad n \geqslant 2.$$

We consider the equation

$$x_{n+1} = F(x_{n-1}, x_{n-2}, x_{n-3}). \tag{2.6}$$

From the conditions of f, g we have that F is a function from $[a_1,b_1] \times [a_1,b_1] \times [a_1,b_1]$ into $[a_1,b_1]$ and $F(x,y,z)$ is increasing in x for all y, z, increasing in y for all x, z and increasing in z for all x, y. Let now that M, m be positive numbers such that

$$M = F(M, M, M) = f(g(M, M), g(M, M)),$$
$$m = F(m, m, m) = f(g(m, m), g(m, m)).$$

By setting $r = g(M,M)$, $R = g(m,m)$ we have that relations Eq. (2.5) are satisfied. Then from hypothesis of Theorem 2.1 we have that $m = M$. Therefore from Theorem 1.15 of [17] we have that Eq. (2.6) has a unique positive equilibrium $\bar{x}$ and every positive solution of Eq. (2.6) tends to the unique positive equilibrium $\bar{x}$. Similarly we can prove that equation

$$y_{n+1} = g(f(y_{n-1},y_{n-2}),f(y_{n-2},y_{n-3})) = G(y_{n-1},y_{n-2},y_{n-3}) \tag{2.7}$$

has a unique positive equilibrium $\bar{y}$ and every positive solution of Eq. (2.7) tends to the unique positive equilibrium $\bar{y}$. This completes the proof of the statement (i).

The proof of the statement (ii) is a modification of similar results included in [10,11,17,24]. □

3. Global behavior of solutions of system Eq. (1.1)

In the first lemma we study the boundedness and persistence of the positive solutions of Eq. (1.1).

Lemma 3.1. *Every positive solution of Eq. (1.1) is bounded and persists.*

Proof. Let (x_n,y_n) be an arbitrary solution of Eq. (1.1). From Eq. (1.1) we can see that

$$x_n \leqslant \frac{\alpha+\beta}{\gamma}, \quad y_n \leqslant \frac{\delta+\epsilon}{\zeta}, \quad n = 1,2,\ldots. \tag{3.1}$$

In addition, from Eqs. (1.1) and (3.1) we get

$$x_n \geqslant \frac{\alpha+\beta e^{-\frac{\delta+\epsilon}{\zeta}}}{\gamma+\frac{\delta+\epsilon}{\zeta}}, \quad y_n \geqslant \frac{\delta+\epsilon e^{-\frac{\alpha+\beta}{\gamma}}}{\zeta+\frac{\alpha+\beta}{\gamma}}, \quad n = 3,4,\ldots \tag{3.2}$$

Therefore, from Eqs. (3.1) and (3.2) the proof of the lemma is complete. □

In the next proposition we will study the asymptotic behavior of the positive solutions of Eq. (1.1).

Proposition 3.1. *Consider system Eq. (1.1). Suppose that the following relation holds true:*

$$\epsilon < \gamma, \quad \beta < \zeta. \tag{3.3}$$

Then system Eq. (1.1) has a unique positive equilibrium $(\bar{x}, \bar{y})$ and every positive solution of Eq. (1.1) tends to the unique positive equilibrium of Eq. (1.1) as $n \to \infty$.

Proof. We consider the functions

$$f(u,v) = \frac{\alpha+\beta e^{-u}}{\gamma+v}, \quad g(z,w) = \frac{\delta+\epsilon e^{-z}}{\zeta+w}, \tag{3.4}$$

where

$$z,w \in I_1 = \left[\frac{\alpha+\beta e^{-\frac{\delta+\epsilon}{\zeta}}}{\gamma+\frac{\delta+\epsilon}{\zeta}}, \frac{\alpha+\beta}{\gamma}\right], \quad u,v \in I_2 = \left[\frac{\delta+\epsilon e^{-\frac{\alpha+\beta}{\gamma}}}{\zeta+\frac{\alpha+\beta}{\gamma}}, \frac{\delta+\epsilon}{\zeta}\right]. \tag{3.5}$$

From Eqs. (3.4) and (3.5), we get the following relations for $u, v \in I_2$, $z, w \in I_1$

$$f(u,v) \in I_1, \quad g(z,w) \in I_2$$

and so $f: I_2 \times I_2 \to I_1$, $g : I_1 \times I_1 \to I_2$. Let (x_n,y_n) be an arbitrary solution of Eq. (1.1). Therefore from Lemma 3.1 for $n \geqslant 3$ we have:

$$x_n \in I_1, \quad y_n \in I_2.$$

Now let m, M, r, R be positive numbers such that

$$M = \frac{\alpha+\beta e^{-r}}{\gamma+r}, \quad m = \frac{\alpha+\beta e^{-R}}{\gamma+R}, \quad R = \frac{\delta+\epsilon e^{-m}}{\zeta+m}, \quad r = \frac{\delta+\epsilon e^{-M}}{\zeta+M}. \tag{3.6}$$

Then we consider the functions

$$F(x) = \frac{\alpha+\beta e^{-h(x)}}{\gamma+h(x)} - x, \quad h(x) = \frac{\delta+\epsilon e^{-x}}{\zeta+x}, \quad x \in I_1. \tag{3.7}$$

Note that F maps the interval I_1 into itself. We claim that equation $F(x) = 0$ has a unique solution in I_1. From Eq. (3.7) we get

$$\begin{aligned} F'(x) &= -h'(x)\tfrac{\beta e^{-h(x)}(\gamma+h(x))+\alpha+\beta e^{-h(x)}}{(\gamma+h(x))^2} - 1, \\ h'(x) &= -\tfrac{\epsilon e^{-x}(\zeta+x)+\delta+\epsilon e^{-x}}{(\zeta+x)^2}, \quad x \in I_1. \end{aligned} \tag{3.8}$$

Let $\bar{x}$, $\bar{x} \in I_1$ be a solution of equation $F(x) = 0$. Then from Eq. (3.7) we have

$$\bar{x}(\gamma + h(\bar{x})) = \alpha + \beta e^{-h(\bar{x})}, \quad h(\bar{x})(\zeta + \bar{x}) = \delta + \epsilon e^{-\bar{x}}. \tag{3.9}$$

Now observe that relations Eqs. (3.8) and (3.9) imply that

$$h'(\bar{x}) = -\frac{\epsilon e^{-\bar{x}} + h(\bar{x})}{\zeta + \bar{x}}, \quad \frac{\beta e^{-h(\bar{x})}(\gamma + h(\bar{x})) + \alpha + \beta e^{-h(\bar{x})}}{(\gamma + h(\bar{x}))^2} = \frac{\beta e^{-h(\bar{x})} + \bar{x}}{\gamma + h(\bar{x})}. \tag{3.10}$$

Then from Eqs. (3.3), (3.8) and (3.10), we get

$$F'(\bar{x}) = \frac{\epsilon e^{-\bar{x}} + h(\bar{x})}{\gamma + h(\bar{x})} \times \frac{\beta e^{-h(\bar{x})} + \bar{x}}{\zeta + \bar{x}} - 1 < 0. \tag{3.11}$$

Therefore, from Eq. (3.11) we see that equation $F(x) = 0$ has a unique solution in I_1. In addition, relations Eq. (3.6) imply that m, M are roots of $F(x) = 0$. Hence we get $m = M$. Therefore from Eq. (3.6) it follows that $r = R$. Thus from statement (i) of Theorem 2.1, system Eq. (1.1) has a unique positive equilibrium $(\bar{x}, \bar{y})$ and every positive solution of system Eq. (1.1) tends to the unique positive equilibrium as $n \to \infty$. This completes the proof of the proposition. □

In the next proposition of this section we will study the global asymptotic stability of the positive equilibrium of Eq. (1.1).

Proposition 3.2. *Consider system Eq. (1.1) where the condition Eq. (3.3) holds true. Also suppose that*

$$\frac{\beta\epsilon + (\beta + \epsilon)e^{-1}}{\gamma\zeta} + \frac{(\alpha + \beta)(\delta + \epsilon)}{\gamma^2\zeta^2} < 1. \tag{3.12}$$

Then the unique positive equilibrium $(\bar{x}, \bar{y})$ of Eq. (1.1) is globally asymptotically stable.

Proof. First we will prove that $(\bar{x}, \bar{y})$ is locally asymptotically stable. The linearized system of Eq. (1.1) about $(\bar{x}, \bar{y})$ is

$$\begin{aligned} x_{n+1} &= -\tfrac{\beta e^{-\bar{y}}}{\gamma+\bar{y}} y_n - \tfrac{\alpha+\beta e^{-\bar{y}}}{(\gamma+\bar{y})^2} y_{n-1} \\ y_{n+1} &= -\tfrac{\epsilon e^{-\bar{x}}}{\zeta+\bar{x}} x_n - \tfrac{\delta+\epsilon e^{-\bar{x}}}{(\zeta+\bar{x})^2} x_{n-1}. \end{aligned} \tag{3.13}$$

Observe that system Eq. (3.13) is equivalent to the following system

$$w_{n+1} = A w_n, \quad A = \begin{pmatrix} 0 & a & 0 & b \\ c & 0 & d & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad w_n = \begin{pmatrix} x_n \\ y_n \\ x_{n-1} \\ y_{n-1} \end{pmatrix},$$

$$a = -\frac{\beta e^{-\bar{y}}}{\gamma + \bar{y}}, \quad b = -\frac{\alpha + \beta e^{-\bar{y}}}{(\gamma + \bar{y})^2}, \quad c = -\frac{\epsilon e^{-\bar{x}}}{\zeta + \bar{x}}, \quad d = -\frac{\delta + \epsilon e^{-\bar{x}}}{(\zeta + \bar{x})^2}.$$

The characteristic equation of A is

$$\lambda^4 - ac\lambda^2 - (ad + bc)\lambda - bd = 0. \tag{3.14}$$

Since $(\bar{x}, \bar{y})$ is the positive equilibrium of Eq. (1.1), then we have

$$\bar{x} = \frac{\alpha + \beta e^{-\bar{y}}}{\gamma + \bar{y}}, \quad \bar{y} = \frac{\delta + \epsilon e^{-\bar{x}}}{\zeta + \bar{x}}. \tag{3.15}$$

Hence from Eqs. (3.12) and (3.15) and since $xe^{-x} < e^{-1}$, $x > 0$ we get

$$\begin{aligned} |ac| + |ad| + |bc| + |bd| &= \frac{\beta\epsilon e^{-\bar{x}-\bar{y}}}{(\gamma + \bar{y})(\zeta + \bar{x})} + \frac{\beta e^{-\bar{y}}(\delta + \epsilon e^{-\bar{x}})}{(\gamma + \bar{y})(\zeta + \bar{x})^2} + \frac{\epsilon e^{-\bar{x}}(\alpha + \beta e^{-\bar{y}})}{(\zeta + \bar{x})(\gamma + \bar{y})^2} + \frac{(\delta + \epsilon e^{-\bar{x}})(\alpha + \beta e^{-\bar{y}})}{(\zeta + \bar{x})^2(\gamma + \bar{y})^2} \\ &= \frac{\beta\epsilon e^{-\bar{x}-\bar{y}}}{(\gamma + \bar{y})(\zeta + \bar{x})} + \frac{\beta\bar{y}e^{-\bar{y}}(\zeta + \bar{x})}{(\gamma + \bar{y})(\zeta + \bar{x})^2} + \frac{\epsilon\bar{x}e^{-\bar{x}}(\gamma + \bar{y})}{(\zeta + \bar{x})(\gamma + \bar{y})^2} + \frac{(\delta + \epsilon e^{-\bar{x}})(\alpha + \beta e^{-\bar{y}})}{(\zeta + \bar{x})^2(\gamma + \bar{y})^2} \\ &< \frac{\beta\epsilon + (\beta + \epsilon)e^{-1}}{\gamma\zeta} + \frac{(\alpha + \beta)(\delta + \epsilon)}{\gamma^2\zeta^2} < 1. \end{aligned} \tag{3.16}$$

Therefore, from Eq. (3.16) and from Remark 1.3.1 of [23], all the roots of Eq. (3.14) are of modulus less than 1 which implies that $(\bar{x}, \bar{y})$ is locally asymptotically stable. Using Proposition 3.1, we see that $(\bar{x}, \bar{y})$ is globally asymptotically stable. This completes the proof of the proposition. □

4. Global Character of solutions of system Eq. (1.2)

In the following lemma we study the boundedness and persistence of system Eq. (1.2).

Lemma 4.1. *Every positive solution of Eq. (1.2) is bounded and persists.*

Proof. Let (x_n, y_n) be an arbitrary solution of Eq. (1.2). Similarly as in lemma 3.1, for $n = 3, 4, \ldots$ by induction we get

$$x_n \in I_3 = \left[\frac{\alpha + \beta e^{-\frac{\delta+\epsilon}{\zeta}}}{\gamma + \frac{\alpha+\beta}{\gamma}}, \frac{\alpha+\beta}{\gamma}\right], \quad y_n \in I_4 = \left[\frac{\delta + \epsilon e^{-\frac{\alpha+\beta}{\gamma}}}{\zeta + \frac{\delta+\epsilon}{\zeta}}, \frac{\delta+\epsilon}{\zeta}\right], \tag{4.1}$$

and so the proof of the lemma is complete. □

In the next proposition we study the asymptotic behavior of the positive solutions of Eq. (1.2).

Proposition 4.1. *Consider system Eq. (1.2). Suppose that the following relation holds true:*

$$\beta\epsilon < \gamma\zeta. \tag{4.2}$$

Then system Eq. (1.2) has a unique positive equilibrium $(\bar{x}, \bar{y})$ and every positive solution of Eq. (1.2) tends to the unique positive equilibrium of Eq. (1.2) as $n \to \infty$.

Proof. We consider the functions

$$f(x, y) = \frac{\alpha + \beta e^{-y}}{\gamma + x}, \quad g(x, y) = \frac{\delta + \epsilon e^{-x}}{\zeta + y}, \tag{4.3}$$

where

$$x \in I_3, \quad y \in I_4, \tag{4.4}$$

and I_3, I_4 are defined in Eq. (4.1). From (4.3) and (4.4), we see that for $x \in I_3$, $y \in I_4$

$$f(x, y) \in I_3, \quad g(x, y) \in I_4$$

and so $f : I_3 \times I_4 \to I_3$, $g : I_3 \times I_4 \to I_4$.

Let m, M, r, R be positive numbers such that

$$M = \frac{\alpha + \beta e^{-r}}{\gamma + m}, \quad m = \frac{\alpha + \beta e^{-R}}{\gamma + M}, \quad R = \frac{\delta + \epsilon e^{-m}}{\zeta + r}, \quad r = \frac{\delta + \epsilon e^{-M}}{\zeta + R}. \tag{4.5}$$

From Eq. (4.5) we get

$$\begin{aligned} e^{-r} &= \tfrac{M(\gamma+m)-\alpha}{\beta}, \quad e^{-R} = \tfrac{m(\gamma+M)-\alpha}{\beta}, \\ e^{-m} &= \tfrac{R(\zeta+r)-\delta}{\epsilon}, \quad e^{-M} = \tfrac{r(\zeta+R)-\delta}{\epsilon}, \end{aligned}$$

which imply that

$$\begin{aligned} M - m &= \tfrac{\beta}{\gamma}(e^{-r} - e^{-R}) = \tfrac{\beta}{\gamma} e^{-r-R}(e^{R} - e^{r}), \\ R - r &= \tfrac{\epsilon}{\zeta}(e^{-m} - e^{-M}) = \tfrac{\epsilon}{\zeta} e^{-m-M}(e^{M} - e^{m}). \end{aligned} \tag{4.6}$$

Moreover, we get

$$\begin{aligned} e^{R} - e^{r} &= e^{\xi}(R - r), \quad \min\{R, r\} \leqslant \xi \leqslant \max\{R, r\}, \\ e^{M} - e^{m} &= e^{\theta}(M - m), \quad \min\{M, m\} \leqslant \theta \leqslant \max\{M, m\}. \end{aligned} \tag{4.7}$$

Then relations Eqs. (4.6) and (4.7) imply that

$$M - m = \frac{\beta}{\gamma} e^{-r-R+\xi}(R - r), \quad R - r = \frac{\epsilon}{\zeta} e^{-m-M+\theta}(M - m)$$

and so

$$|M - m| \leqslant \frac{\beta}{\gamma}|R - r|, \quad |R - r| \leqslant \frac{\epsilon}{\zeta}|M - m|. \tag{4.8}$$

In addition, observe that relations Eqs. (4.2) and (4.8) imply that

$$\left(1 - \frac{\beta\epsilon}{\gamma\zeta}\right)|M - m| \leqslant 0, \quad \left(1 - \frac{\beta\epsilon}{\gamma\zeta}\right)|R - r| \leqslant 0$$

from which we see that $M = m$ and $R = r$. Therefore from Eq. (4.1) and statement (ii) of Theorem 2.1 system Eq. (1.2) has a unique positive equilibrium $(\bar{x},\ \bar{y})$ and every positive solution of system Eq. (1.2) tends to the unique positive equilibrium as $n \to \infty$. This completes the proof of the proposition. □

In the next proposition of this section we will study the global asymptotic stability of the positive equilibrium of Eq. (1.2).

Proposition 4.2. *Consider system Eqs. (1.2) where (4.2) holds true. Also suppose that*

$$\frac{\alpha+\beta}{\gamma^2}+\frac{\delta+\epsilon}{\zeta^2}+\frac{\beta\epsilon}{\gamma\zeta}+\frac{(\alpha+\beta)(\delta+\epsilon)}{\gamma^2\zeta^2}<1. \tag{4.9}$$

Then the unique positive equilibrium $(\bar{x},\ \bar{y})$ of Eq. (1.2) is globally asymptotically stable.

Proof. First we will prove that $(\bar{x},\ \bar{y})$ is locally asymptotically stable. The linearized system of Eq. (1.2) about $(\bar{x},\ \bar{y})$ is

$$\begin{aligned} x_{n+1} &= -\tfrac{\beta e^{-\bar{y}}}{\gamma+\bar{x}} y_n - \tfrac{\alpha+\beta e^{-\bar{y}}}{(\gamma+\bar{x})^2} x_{n-1} \\ y_{n+1} &= -\tfrac{\epsilon e^{-\bar{x}}}{\zeta+\bar{y}} x_n - \tfrac{\delta+\epsilon e^{-\bar{x}}}{(\zeta+\bar{y})^2} y_{n-1}. \end{aligned} \tag{4.10}$$

Notice that system Eq. (4.10) is equivalent to the following system

$$w_{n+1} = Bw_n, \quad B = \begin{pmatrix} 0 & a & b & 0 \\ c & 0 & 0 & d \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad w_n = \begin{pmatrix} x_n \\ y_n \\ x_{n-1} \\ y_{n-1} \end{pmatrix},$$

$$a = -\frac{\beta e^{-\bar{y}}}{\gamma+\bar{x}}, \quad b = -\frac{\alpha+\beta e^{-\bar{y}}}{(\gamma+\bar{x})^2}, \quad c = -\frac{\epsilon e^{-\bar{x}}}{\zeta+\bar{y}}, \quad d = -\frac{\delta+\epsilon e^{-\bar{x}}}{(\zeta+\bar{y})^2}.$$

Then the characteristic equation of B is

$$\lambda^4 - (b+d+ac)\lambda^2 + bd = 0. \tag{4.11}$$

From Eq. (4.9) we get

$$\begin{aligned} |b|+|d|+|ac|+|bd| &= \frac{\alpha+\beta e^{-\bar{y}}}{(\gamma+\bar{x})^2}+\frac{\delta+\epsilon e^{-\bar{x}}}{(\zeta+\bar{y})^2}+\frac{\beta\epsilon e^{-\bar{x}-\bar{y}}}{(\gamma+\bar{x})(\zeta+\bar{y})}+\frac{(\delta+\epsilon e^{-\bar{x}})(\alpha+\beta e^{-\bar{y}})}{(\zeta+\bar{y})^2(\gamma+\bar{x})^2} \\ &< \frac{\alpha+\beta}{\gamma^2}+\frac{\delta+\epsilon}{\zeta^2}+\frac{\beta\epsilon}{\gamma\zeta}+\frac{(\alpha+\beta)(\delta+\epsilon)}{\gamma^2\zeta^2}<1. \end{aligned} \tag{4.12}$$

Therefore, from Eq. (4.12) and from Remark 1.3.1 of [23] all the roots of Eq. (4.11) are of modulus less than 1 which implies that $(\bar{x},\ \bar{y})$ is locally asymptotically stable. Using Proposition 4.1, we see that $(\bar{x},\ \bar{y})$ is globally asymptotically stable. This completes the proof of the proposition. □

5. Global character of solutions of system Eq. (1.3)

In the following lemma we study the boundedness and persistence of system Eq. (1.3).

Lemma 5.1. *Every positive solution of Eq. (1.3) is bounded and persists.*

Proof. Let (x_n, y_n) be an arbitrary solution of Eq. (1.3). Similarly as in lemma 3.1, for $n = 3,4,\ldots$ we get

$$x_n \in I_5 = \left[\frac{\alpha+\beta e^{-\frac{\alpha+\beta}{\gamma}}}{\gamma+\frac{\delta+\epsilon}{\zeta}}, \frac{\alpha+\beta}{\gamma}\right], \quad y_n \in I_6 = \left[\frac{\delta+\epsilon e^{-\frac{\delta+\epsilon}{\zeta}}}{\zeta+\frac{\alpha+\beta}{\gamma}}, \frac{\delta+\epsilon}{\zeta}\right], \tag{5.1}$$

and so the proof of the lemma is complete. □

In the next proposition we study the asymptotic behavior of the positive solutions of Eq. (1.3).

Proposition 5.1. *Consider system Eq. (1.3). Suppose that the following relation holds true:*

$$\beta < \gamma, \quad \epsilon < \zeta. \tag{5.2}$$

Then system Eq. (1.3) has a unique positive equilibrium $(\bar{x}, \bar{y})$ and every positive solution of Eq. (1.3) tends to the unique positive equilibrium of Eq. (1.3) as $n \to \infty$.

Proof. We consider the functions

$$f(x,y) = \frac{\alpha + \beta e^{-x}}{\gamma + y}, \quad g(x,y) = \frac{\delta + \epsilon e^{-y}}{\zeta + x}, \tag{5.3}$$

where

$$x \in I_5, \quad y \in I_6, \tag{5.4}$$

where I_5, I_6 are defined in Eq. (5.1). From Eq. (5.3) and Eq. (5.4), we have for $x \in I_5$, $y \in I_6$

$$f(x,y) \in I_5, \quad g(x,y) \in I_6$$

and so $f : I_5 \times I_6 \to I_5$, $g : I_5 \times I_6 \to I_6$.

Let m, M, r, R be positive numbers such that

$$M = \frac{\alpha + \beta e^{-m}}{\gamma + r}, \quad m = \frac{\alpha + \beta e^{-M}}{\gamma + R}, \quad R = \frac{\delta + \epsilon e^{-r}}{\zeta + m}, \quad r = \frac{\delta + \epsilon e^{-R}}{\zeta + M}. \tag{5.5}$$

Moreover arguing as in the proof of Theorem 1.16 of [17], it suffices to assume that

$$m \leqslant M, \quad r \leqslant R. \tag{5.6}$$

Observe that relations Eq. (5.5) imply:

$$e^{-m} = \tfrac{M(\gamma+r)-\alpha}{\beta}, \quad e^{-M} = \tfrac{m(\gamma+R)-\alpha}{\beta},$$
$$e^{-r} = \tfrac{R(\zeta+m)-\delta}{\epsilon}, \quad e^{-R} = \tfrac{r(\zeta+M)-\delta}{\epsilon}$$

from which we see that

$$\epsilon(e^{-r} - e^{-R}) = \zeta(R - r) + Rm - rM, \quad \beta(e^{-m} - e^{-M}) = \gamma(M - m) + Mr - mR. \tag{5.7}$$

Then by adding the two relations Eq. (5.7) we get

$$\epsilon(e^{-r} - e^{-R}) + \beta(e^{-m} - e^{-M}) = \zeta(R - r) + \gamma(M - m).$$

Thus from Eq. (4.7) we get

$$\epsilon e^{-r-R+\xi}(R - r) + \beta e^{-m-M+\theta}(M - m) = \zeta(R - r) + \gamma(M - m) \tag{5.8}$$

where θ, ξ are defined in Eq. (4.7). Therefore from Eq. (5.8) we have

$$\zeta(R - r)\left(1 - \frac{\epsilon}{\zeta}e^{-r-R+\xi}\right) + \gamma(M - m)\left(1 - \frac{\beta}{\gamma}e^{-m-M+\theta}\right) = 0. \tag{5.9}$$

Then using Eqs. (5.2), (5.6) and (5.9), gives us $m = M$ and $r = R$. Hence from Eq. (5.1), statement (ii) of Theorem 2.1 system Eq. (1.3) has a unique positive equilibrium $(\bar{x}, \bar{y})$ and every positive solution of system Eq. (1.3) tends to the unique positive equilibrium as $n \to \infty$. This completes the proof of the proposition. □

In the next proposition of this paper, we study the global asymptotic stability of the positive equilibrium of Eq. (1.3).

Proposition 5.2. *Consider system Eqs. (1.3) where (5.2) hold true. Also suppose that*

$$\frac{\beta}{\gamma} + \frac{\epsilon}{\zeta} + \frac{\beta\epsilon}{\gamma\zeta} + \frac{(\alpha + \beta)(\delta + \epsilon)}{\gamma^2\zeta^2} < 1. \tag{5.10}$$

Then the unique positive equilibrium $(\bar{x}, \bar{y})$ of Eq. (1.3) is globally asymptotically stable.

Proof. First we will prove that $(\bar{x}, \bar{y})$ is locally asymptotically stable. The linearized system of Eq. (1.3) about $(\bar{x}, \bar{y})$ is

$$\begin{aligned} x_{n+1} &= -\tfrac{\beta e^{-\bar{x}}}{\gamma+\bar{y}} x_n - \tfrac{\alpha+\beta e^{-\bar{x}}}{(\gamma+\bar{y})^2} y_{n-1}, \\ y_{n+1} &= -\tfrac{\epsilon e^{-\bar{y}}}{\zeta+\bar{x}} y_n - \tfrac{\delta+\epsilon e^{-\bar{y}}}{(\zeta+\bar{x})^2} x_{n-1}. \end{aligned} \tag{5.11}$$

Clearly we see that system Eq. (5.11) is equivalent to the system

$$w_{n+1} = Cw_n, \quad C = \begin{pmatrix} a & 0 & 0 & b \\ 0 & c & d & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad w_n = \begin{pmatrix} x_n \\ y_n \\ x_{n-1} \\ y_{n-1} \end{pmatrix},$$

$$a = -\frac{\beta e^{-\bar{x}}}{\gamma + \bar{y}}, \quad b = -\frac{\alpha + \beta e^{-\bar{x}}}{(\gamma + \bar{y})^2}, \quad c = -\frac{\epsilon e^{-\bar{y}}}{\zeta + \bar{x}}, \quad d = -\frac{\delta + \epsilon e^{-\bar{y}}}{(\zeta + \bar{x})^2}.$$

Then the characteristic equation of C is

$$\lambda^4 - (a + c)\lambda^3 + ac\lambda^2 - bd = 0. \tag{5.12}$$

From Eq. (5.10) we get

$$|a| + |c| + |ac| + |bd| = \frac{\beta e^{-\bar{x}}}{\gamma + \bar{y}} + \frac{\epsilon e^{-\bar{y}}}{\zeta + \bar{x}} + \frac{\beta\epsilon e^{-\bar{x}-\bar{y}}}{(\gamma + \bar{y})(\zeta + \bar{x})} + \frac{(\delta + \epsilon e^{-\bar{y}})(\alpha + \beta e^{-\bar{x}})}{(\zeta + \bar{x})^2(\gamma + \bar{y})^2} < \frac{\beta}{\gamma} + \frac{\epsilon}{\zeta} + \frac{\beta\epsilon}{\gamma\zeta} + \frac{(\alpha + \beta)(\delta + \epsilon)}{\gamma^2\zeta^2} < 1. \tag{5.13}$$

Therefore, from Eq. (5.13) and from Remark 1.3.1 of [23] we see that all the roots of Eq. (5.12) are of modulus less than 1, which implies that $(\bar{x}, \bar{y})$ is locally asymptotically stable. Using Proposition 5.1, $(\bar{x}, \bar{y})$ is globally asymptotically stable. This completes the proof of the proposition. □

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