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Spectral Analysis Techniques of Stationary Point Processes: Extensions and Applications to Neurophysiological Problems

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Abstract—In this work we examine the behaviour of a complex physiological system (muscle spindle) by using spectral analysis techniques of stationary point processes. In particular, we investigate the effect of a gamma motoneuron on the complex system when

(a) there is no other stimulus present, and
(b) there is an alpha motoneuron present.

It is shown that the presence of an alpha motoneuron reduces the effect of the gamma motoneuron on the muscle spindle.

Keywords—Stationary point processes, Spectral density function, Coherence function, Cross-intensity function, Muscle spindle.

1. INTRODUCTION

The mathematical problem presented in this paper is related with the neurophysiological system called muscle spindle and its behaviour to certain practical situations. As we shall see in the next section, the behaviour of the muscle spindle can be modified by the effect of the alpha and gamma motoneurons whose bodies lie inside the spinal cord and make synaptic contacts with interneurons and the axons from higher levels in central nervous system. When the muscle spindle is not affected by any stimulus, its response is quite regular (the distances between the nerve pulses recorded from the muscle spindle are almost the same). This regularity is destroyed completely by the effect of a gamma motoneuron (see [1]). Here, we shall investigate the effect of a gamma motoneuron on the muscle spindle when the effect of an alpha motoneuron is present as well. A technique of spectral analysis for a bivariate stationary point process is used in order to examine how the incoming information to the muscle spindle is correlated with the outgoing information directed towards the spinal cord by estimating certain parameters of the bivariate point process in the frequency domain. By getting the inverse Fourier transform, we are able to estimate certain parameters of the bivariate point process and develop its asymptotic properties in the time domain, and hence, to be in a position to obtain useful information about the behaviour of the muscle spindle which is in agreement with the results of the frequency domain.

2. BRIEF DESCRIPTION OF THE MUSCLE SPINDLE

The muscle spindle is an element of the neuromuscular system and plays a critical role in the initiation of movement and the maintenance of posture. It is also a transducer which responds to different stimuli applied on it. Most skeletal muscles contain a number of these transducers,

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which lie in parallel with the fibers of the muscle (known as extrafusal fibers). The fibers within a muscle spindle, known as intrafusal fibers, are considerably shorter than the extrafusal fibers. There are three different types of intrafusal muscle fibers, the dynamical nuclear-bag (DNB), the static nuclear-bag (SNB) and the nuclear chain (NC).

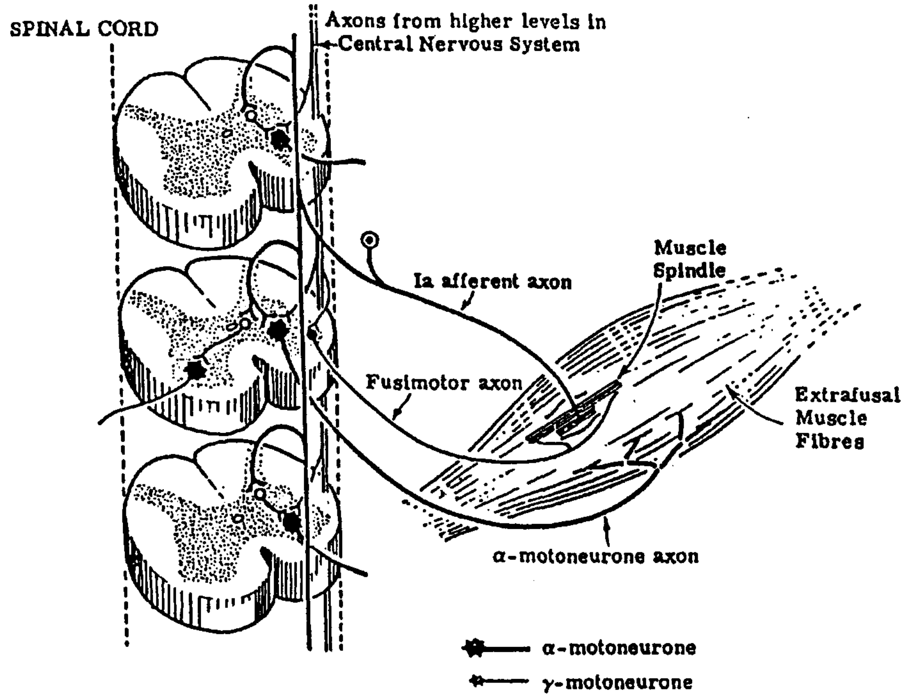


Figure 1. Diagram showing the connections between a muscle spindle, its parent muscle and the spinal cord.

The effect of a stimulus on the muscle spindle is transmitted to the spinal cord by the terminal branches of the axons of sensory neurons which are wrapped round all of the intrafusal fibers. Figure 1 shows the muscle spindle, its parent muscle and how the response from the muscle spindle is transmitted to the spinal cord through the axon of a sensory neuron called Ia afferent axon. We can also see how a gamma motoneuron affects directly the muscle spindle by making synaptic contact with the intrafusal fibers and how an alpha motoneuron affects indirectly the muscle spindle by making synaptic contact with the extrafusal fibres. More details about the muscle spindle are given in [2].

3. SPECTRAL ANALYSIS OF STATIONARY POINT PROCESSES

The muscle spindle can be assumed as a stochastic system involving point processes. By this we mean that the input to and the output of the system are point processes. Mathematically, a point process is defined as a random, nonnegative, integer-valued measure.

Let $\{N_1(t), N_2(t)\}$, $-\infty < t < \infty$, be a bivariate point process which is assumed to be stationary, orderly and strong mixing. These assumptions are satisfied approximately in practice and discussed in detail by Brillinger [3] and Daley and Vere-Jones [4].

We must stress at this point that our analysis will be based on methods of the frequency domain in order to obtain estimates of certain parameters of the bivariate point process and extract useful information about the behaviour of the muscle spindle. The second-order spectral density of a stationary point process is defined by

$$f_{ab}(\lambda) = (2\pi)^{-1} \left[q_a \delta\{a-b\} + \int_{-\infty}^{+\infty} \exp\{-i\lambda u\}\, q_{ab}(u)\, du \right], \qquad -\infty < \lambda < \infty, \tag{1}$$

where q_a is the mean-intensity of the component a, $q_{ab}(u)$ is the cumulant density and $\delta\{u\}$ is the Kronecker delta $(a, b = 1, 2)$. By getting the inverse Fourier transform of (1), we find

$$q_{ab}(u) = \int_{-\infty}^{+\infty} \exp(i\lambda u)\left[f_{ab}(\lambda) - \frac{q_a}{2\pi}\delta\{a-b\}\right] d\lambda. \tag{2}$$

Higher order spectral and cumulant densities of stationary point processes can also be defined. For more details refer to [3].

Another useful function is the intensity function which is defined by

$$m_{ab}(u) = \frac{q_{ab}(u)}{q_b} + q_a, \qquad (a, b = 1, 2). \tag{3}$$

This function is a conditional probability and can be interpreted as

$$\text{Prob}\{\text{event of } a\text{-type at } (t+u, t+u+h]/\text{event of } b\text{-type at } t\},$$

where h is small.

4. ESTIMATION OF THE PARAMETERS

We consider that the stationary bivariate point process is observed on the time interval $(0, T]$. In order to obtain an estimate of the second-order spectral density we split the whole record of the data T into L disjoint subrecords each of length R, i.e., $T = LR$. In each subrecord, we compute the periodogram statistic and then we find an estimate of the spectral density by averaging the periodogram ordinates in every frequency. The periodogram of the j^{th} subrecord is defined by

$$I_{ab}^{(R)}(\lambda, j) = (2\pi R)^{-1} d_a^{(R)}(\lambda, j)\,\overline{d_b^{(R)}(\lambda, j)} \qquad \text{for } \lambda \neq 0,\ j = 0, 1, \ldots, L-1 \quad (a, b = 1, 2), \tag{4}$$

where $d_a^{(R)}(\lambda, j)$ is the finite Fourier-Stieltjes transform of the j^{th} subrecord given by

$$d_a^{(R)}(\lambda, j) = \int_{jR}^{(j+1)R} \exp\{-i\lambda t\}\, dN_a(t), \qquad (a = 1, 2). \tag{5}$$

By $\overline{d_b^{(R)}(\lambda, j)}$ we denote the conjugate function of $d_b^{(R)}(\lambda, j)$ and by $dN_a(t) = N_a(t, t+dt]$ the number of events of a-type which occur in the interval $(t, t+dt]$. An estimate of the spectral density can now be obtained by

$$f_{ab}^{(LR)}(\lambda) = L^{-1} \sum_{j=0}^{L-1} I_{ab}^{(R)}(\lambda, j), \qquad \text{for } \lambda \neq 0, \quad (a, b = 1, 2). \tag{6}$$

We can further improve the properties of this estimate by applying the following weighting scheme

$$\hat{f}_{ab}^{(LR)}(\lambda_k) = \frac{1}{2p+1} \sum_{r=-p}^{p} f_{ab}^{(LR)}(\lambda_{k+r}), \quad \text{where } \lambda_k = \frac{2\pi k}{R} \text{ and } k = 1, 2, \ldots, \frac{R-1}{2}. \tag{7}$$

In order to test whether the components of the bivariate point process are correlated, we use the coherence function, an estimate of which is given by

$$\left|\hat{R}_{21}(\lambda)\right|^2 = \frac{\left|\hat{f}_{21}^{(LR)}(\lambda)\right|^2}{\hat{f}_{11}^{(LR)}(\lambda)\hat{f}_{22}^{(LR)}(\lambda)}, \qquad \text{for } \lambda \neq 0. \tag{8}$$

A 100α percent point of the estimate of the coherence function is obtained from the following relation

$$z = 1 - (1-\alpha)^{1/s-1} \tag{9}$$

where $s = (2p+1)L$. Values of the estimate close or below z infer that the components of the bivariate process are uncorrelated. More details about the estimation of the spectral density and the coherence function can be found in [1,5].

We proceed now to obtain estimates of the cross-cumulant density and the cross-intensity function based on the estimate of the cross-spectral density.

An estimate of the cross-cumulant density is given by

$$\hat{q}_{ab}(u) = \frac{2\pi}{Q_R} \sum_p \hat{f}_{ab}^{(LR)}(\lambda_p) \exp(i\lambda_p u), \qquad p = 0, 1, \ldots, \frac{Q_R - 1}{2}, \quad (a \neq b). \tag{10}$$

The quantity Q_R is chosen in such a way that $Q_R \to \infty$ as $R \to \infty$. Thus, an estimate of the cross-intensity function can also be obtained from (3) as follows:

$$\hat{m}_{ab}(u) = \frac{\hat{q}_{ab}(u)}{\hat{q}_b} + \hat{q}_a, \tag{11}$$

where $\hat{q}_a = (N_a(t))/T\,(a = 1,2)$ is the estimate of the mean intensity. We examine now the asymptotic properties of the estimate of the cross-intensity function.

THEOREM 1. *Let $\{N_1(t), N_2(t)\}$, $-\infty < t < \infty$, be a stationary bivariate point process. We also assume that the process is orderly, mixing and its moments are finite. Then, if $Q_R \to \infty$ and $Q_R R^{-1} \to 0$ as $R \to \infty$, the estimate $\hat{m}_{ab}(u)$ is asymptotically normal with mean $m_{ab}(u)$ and variance*

$$\lim_{R\to\infty} Q_R \mathrm{Var}[\hat{m}_{ab}(u)] = \frac{2\pi}{(2p+1)Lq_b^2} \left[\int f_{aa}(\lambda) f_{bb}(\lambda)\, d\lambda + \int f_{ab}(\lambda) f_{ba}(-\lambda) \exp(i\lambda 2u)\, d\lambda \right].$$

In order to improve the properties of $\hat{m}_{ab}(u)$, we insert a convergence factor in the estimate of the cross-cumulant density as follows:

$$\tilde{q}_{ab}(u) = \frac{2\pi}{Q_R} \sum_p W_R(\lambda_p) \hat{f}_{ab}^{(LR)}(\lambda_p) \exp(i\lambda_p u), \tag{12}$$

where $W_R(\lambda) = W(b_R\lambda)$ is the convergence factor. The quantity b_R is the bandwidth which is chosen in such a way that $b_R \to 0$ as $R \to \infty$. For more details about convergence factors, refer to [6].

The asymptotic properties of the new estimate of the cross-intensity function, $\tilde{m}_{ab}(u)$, are examined in the next corollary.

COROLLARY 1. *We suppose that the bivariate point process satisfies the assumptions of Theorem 1. Then, if $b_R Q_R \to \infty$ as $R \to \infty$, the estimate $\tilde{m}_{ab}(u)$ is asymptotically normal with mean q_a and variance*

$$\lim_{R\to\infty} b_R Q_R \mathrm{Var}[\tilde{m}_{ab}(u)] = \frac{(2\pi)^{-1} q_a}{(2p+1)Lq_b} \int W^2(\lambda)\, d\lambda. \tag{13}$$

The Theorem and the Corollary are extensions of the results developed in [5] and their proofs can be obtained in a similar way.

It follows from Corollary 1 that a 95% approximate confidence interval for $\tilde{m}_{ab}(u)$ will be given by

$$\hat{q}_a \pm 1.96 \left[\frac{(2\pi)^{-1} \hat{q}_a b_R^{-1}}{(2p+1)L\hat{q}_b Q_R} \int W^2(\lambda)\, d\lambda \right]^{1/2}. \tag{14}$$

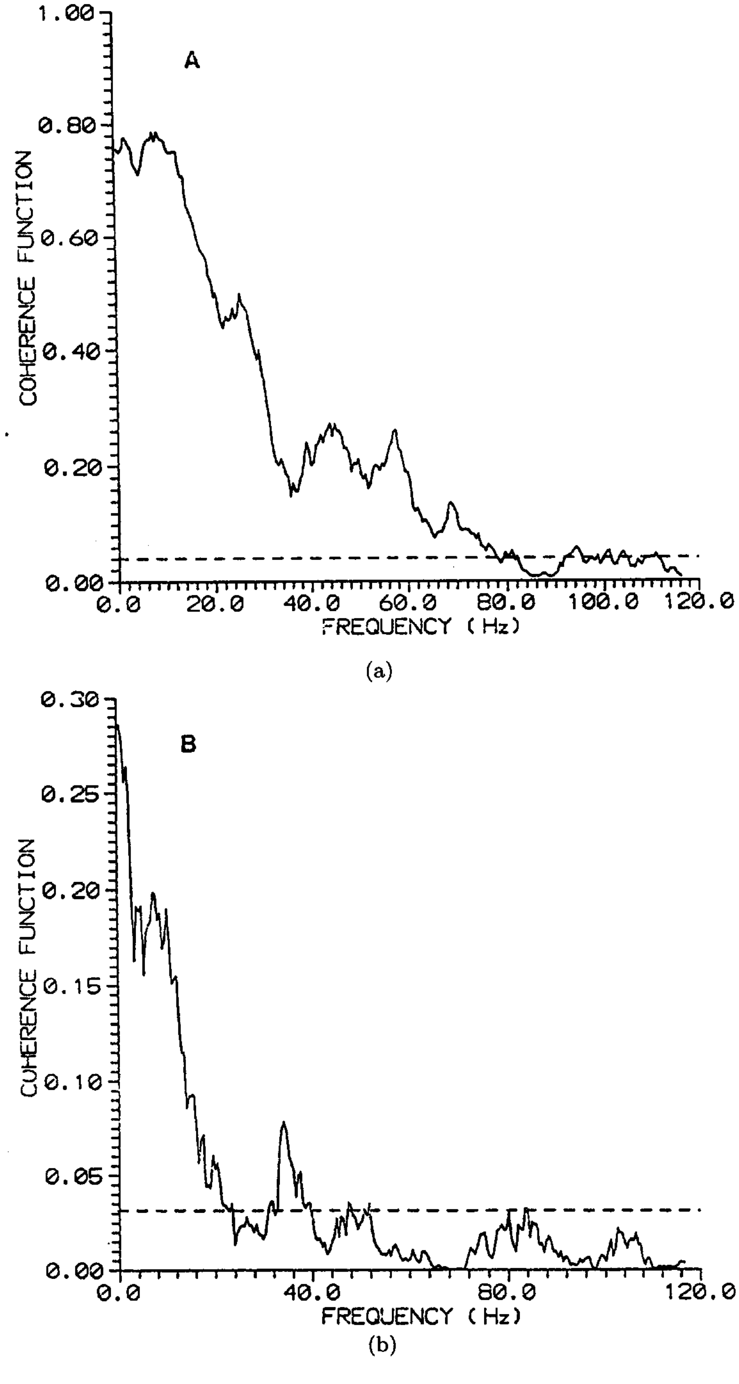


Figure 2. Estimates of the coherence function when the muscle spindle is affected by a gamma motoneuron and simultaneously (a) there is no other stimulus present, (b) there is an alpha motoneuron present.

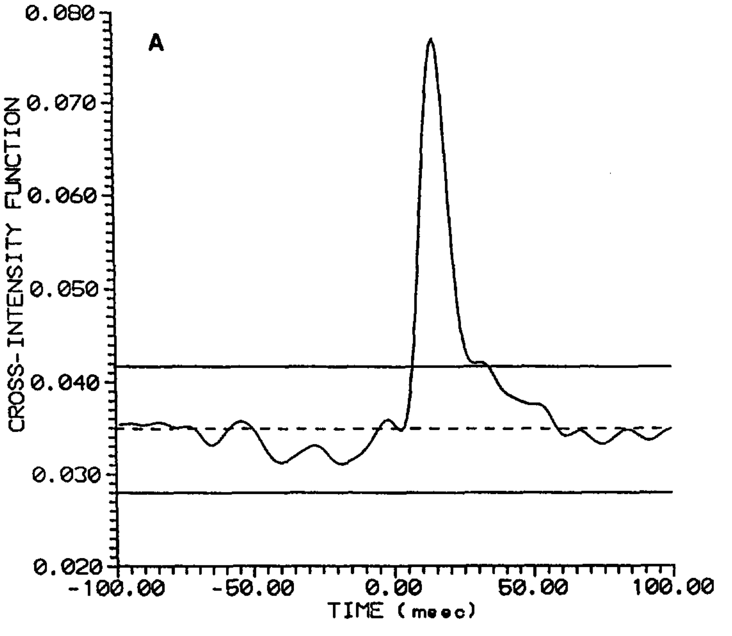
5. EXAMPLES

We apply now the results of the analysis described previously to investigate the behaviour of the muscle spindle when it is affected by a gamma motoneuron and at the same time

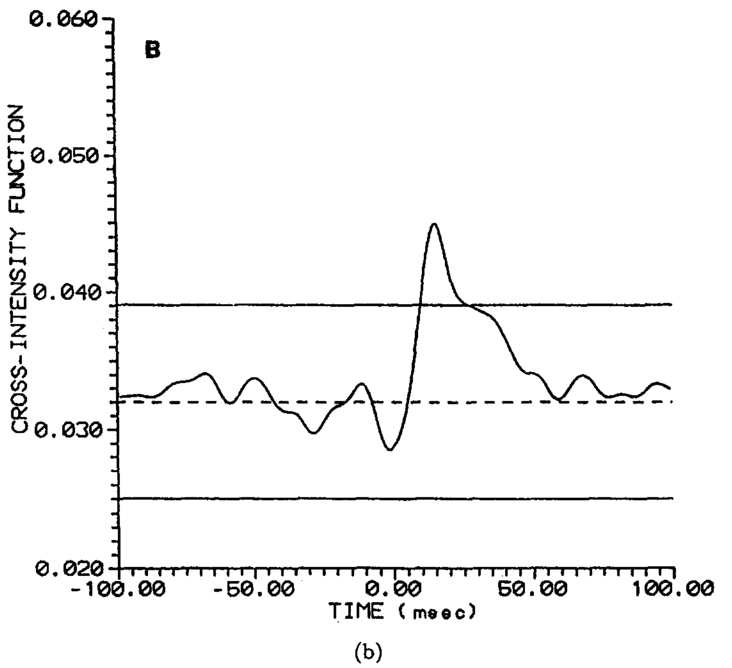
(a) there is no other stimulus present, and
(b) the effect of an alpha motoneuron is present.

For the estimation of the spectral density function the whole record of the data $T = 10240$ msec was divided into $L = 5$ disjoint subrecords of length $R = 2048$ msec each. In order to improve further this estimate we have used $p = 9$ in (7). The number of events recorded in the time interval $T = 10240$ msec for the input to and the output of the muscle spindle were as follows:

(a) $N_1(T) = 661$, $N_2(T) = 357$, and
(b) $N_1(T) = 617$, $N_2(T) = 357$.



(a)



(b)

Figure 3. Estimates of the cross-intensity function for the cases (a) and (b) described in Figure 2 when the muscle spindle is affected by a gamma motoneuron.

Figure 2 shows the estimates of the coherence function for the two cases (a) and (b). The dotted lines in the figures indicate the 95% confidence limits obtained from (9) where $a = 0.95$ and $s = (2p+1)L = 95$. It is obvious from Figure 2b that the values of the estimated coherence function have been reduced considerably compared with those of Figure 2a. Also the range of frequencies in which the two point process are correlated has been reduced from (0–80) Hz to (0–40) Hz. These results suggest that the presence of the alpha motoneuron reduces considerably the effect of the gamma motoneuron on the muscle spindle.

Figure 3 presents the estimates of the cross-intensity function for the two cases (a) and (b). The dotted lines in the middle of the figures correspond to the mean values of the estimates and the solid lines correspond to the 95% confidence limits of the estimates obtained from expression (14) by setting $Q_R = 256$, $b_R = 30$ and using a Tukey convergence factor. From Figure 2b becomes clear that the values of the estimate of the conditional probability (interpretation of the cross-intensity function) have been reduced considerably compared to those of the estimate of Figure 2a. This result again suggests that the presence of the alpha motoneuron reduces considerably the effect of the gamma motoneuron on the muscle spindle.

6. CONCLUSIONS

We have developed a technique of spectral analysis for stationary point processes in order to study the behaviour of the neurophysiological system muscle spindle. Estimates of the coherence function and the cross-intensity function are based on the estimate of the spectral density function. These estimates are obtained by analysing two data sets which correspond to the cases where the muscle spindle is affected by a gamma motoneuron and simultaneously,

(a) no other stimulus is present, and
(b) an alpha motoneuron is present.

It is shown that the presence of an alpha motoneuron reduces considerably the effect of the gamma motoneuron on the muscle spindle.

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