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Estimation of Certain Parameters of a Stationary Hybrid Process Involving a Time Series and a Point Process

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Received 22 August 1994; revised 6 June 1995

ABSTRACT

A method is presented for estimating the cross-spectral density of a hybrid process involving a time series and a point process. The method is based on the generalized cross-periodogram statistic, which is smoothed by splitting the whole record of the data into a number of disjoint subrecords. Estimates of the coherence function and the cross-covariance function can also be obtained by using the estimate of the cross-spectral density. The distribution of the cross-covariance function between a time series and a point process is shown to be asymptotically normal. The theoretical results are used in the study of a complex physiological system. It is shown that the presence of a gamma motor neuron (gamma stimulation) modifies the effect of the length changes on the complex system at low frequencies (the length changes, and the response of the system become uncorrelated in the range 3–30 Hz) while the effect remains unchanged at higher frequencies. As a comparison it is shown that the presence of the length changes weakens the effect of the gamma stimulation on the complex system.

1. INTRODUCTION

In practical problems there are systems that involve a time series and a point process (see [1–5]). To be able to estimate certain parameters of such hybrid processes and thus to reach some useful conclusions we need to extend known results from the theory of point processes and time series.

Our approach in this work to handle such a problem is based on a technique of spectral analysis for a hybrid process involving a time series and a point process. We generalize the definition of the cross-periodogram statistic for the case of a hybrid process, and by smoothing it we obtain an estimate of the cross-spectral density. The smoothing of

MATHEMATICAL BIOSCIENCES 133:197–218 (1996)

655 Avenue of the Americas, New York, NY 10010 0025-5564/96/$15.00
SSDI 0025-5564(95)00106-N

the cross-periodogram is done by dividing the whole record of the data into a number of disjoint subrecords.

Once we have obtained an estimate of the cross-spectral density we can proceed to find estimates of certain parameters of the hybrid process in both the time and frequency domains. An estimate of the coherence function provides a measure of the linear time-invariant relationship between a time series and a point process in the frequency domain, while an estimate of the cross-covariance function shows the relation between a time series and a point process in the time domain. An estimate of the cross-covariance function is obtained by estimating the inverse Fourier transform of the cross-spectral density. Willie [5] discusses estimates of certain time domain parameters and develops their asymptotic properties. These estimates can also be obtained by using the estimate of the cross-covariance function.

One field of research where this kind of problem arises is that of neurophysiology. A complex physiological system called the muscle spindle responds to length changes imposed on the parent muscle. In this case the muscle spindle is assumed to be a hybrid system involving a time series (changes in the length of the parent muscle) and a point process (response of the muscle spindle). By analyzing two data sets recorded from neurophysiological experiments we examine the behavior of the muscle spindle when it is affected by length changes imposed on the parent muscle and at the same time (A) there is no other stimulus present or (B) a gamma stimulation is present. It is shown that the presence of the gamma stimulation has an effect on the muscle spindle at low frequencies but at higher frequencies the effect remains unchanged.

Finally, by estimating the cross-cumulant of a stationary bivariate point process we show that the presence of length changes on the parent muscle weakens the effect of gamma stimulation on the muscle spindle.

2. AN ESTIMATE OF THE CROSS-SPECTRAL DENSITY AND ITS ASYMPTOTIC PROPERTIES

An estimate of the cross-spectral density of a hybrid process can be obtained by generalizing the definition of the cross-periodogram statistic between a time series and a point process. Let $\{X(t), N(t)\}$, $-\infty < t < +\infty$, be a hybrid process consisting of a time series and a point process. It will be assumed that the hybrid process satisfies the following assumptions:

(1) It is stationary. This means that the probabilistic structure of the process does not change with time.

(2) It is strong mixing. This means that the increments of $\{N(\Delta), Y(\Delta)\}$ well separated in time become independent, where $N(\Delta) = \int_\Delta dN(t)$, $Y(\Delta) = \int_\Delta X(t)\,dt$, and $\Delta = (s, t]$.

(3) The time series $\{X(t)\}$ and the point process $\{N(t)\}$ are jointly stationary processes.

(4) The point process $\{N(t)\}$ is orderly. This means that the probability of having two or more events in a small interval is negligible.

More details about these assumptions can be found in Brillinger [6, 7], Daley and Vere-Jones [8], Willie [5], and Rigas [9].

Before we go on to define the cross-periodogram statistic between a time series and a point process it is necessary to present some parameters of the hybrid process in both the time and frequency domains. In the time domain we can define the moments and the covariance functions.

The second-order moment between a time series and a point process is defined by

$$\mu_{NX}(u)\,du = E\{dN(t+u)X(t)\}, \tag{1}$$

where $dN(t)$ is the increment of the point process in the time interval $(t, t+dt]$. The third-order moments are defined as

$$\mu_{NXX}(u,v)\,du = E\{dN(t+u)X(t+v)X(t)\} \tag{2}$$

and

$$\begin{aligned}[\mu_{NNX}(u,v) + \mu_{NX}(u)\,\delta(u-v)]\,du\,dv \\ = E\{dN(t+u)\,dN(t+v)X(t)\}.\end{aligned} \tag{3}$$

In a similar way we can define higher order moments of the hybrid process.

The cross-covariance function between a time series and a point process is defined by

$$c_{NX}(u)\,du = \operatorname{cov}\{dN(t+u), X(t)\}. \tag{4}$$

In the same way we can define higher order covariance functions. For example, the third-order covariance function $c_{NXX}(u,v)$ is given by

$$c_{NXX}(u,v)\,du = \operatorname{cum}\{dN(t+u), X(t+v), X(t)\}. \tag{5}$$

In the frequency domain we define the cross-spectral density as

$$f_{NX}(\lambda) = (2\pi)^{-1} \int_{-\infty}^{+\infty} c_{NX}(u) \exp\{-i\lambda u\}\, du, \qquad -\infty < \lambda < +\infty, \quad (6)$$

provided that $\int_{-\infty}^{+\infty} |c_{NX}(u)|\, du < +\infty$.

The third-order spectral density of $\{N, X\}$ is defined by

$$\begin{aligned} f_{NXX}(\lambda, \mu) = (2\pi)^{-2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} & c_{NXX}(u, v) \\ & \times \exp\{-i(\lambda u + \mu v)\}\, du\, dv, \qquad -\infty < \lambda, \mu < +\infty, \end{aligned} \quad (7)$$

provided that $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |c_{NXX}(u, v)|\, du\, dv < +\infty$. In a similar way we can define spectral densities of higher order.

We can proceed now to define the cross-periodogram statistic for a hybrid process. We assume that the process $\{X(t), N(t)\}$ is observed on $(0, R]$. Then the cross-periodogram statistic is given by

$$I_{NX}^{(R)}(\lambda) = \frac{1}{2\pi R} d_N^{(R)}(\lambda)\, \overline{d_X^{(R)}(\lambda)}, \qquad -\infty < \lambda < +\infty, \quad (8)$$

where $\overline{d_X^{(R)}(\lambda)}$ is the conjugate function of $d_X^{(R)}(\lambda)$. We denote the finite Stieltjes–Fourier transform of the point process by $d_N^{(R)}(\lambda)$, and the finite Fourier transform of the time series by $d_X^{(R)}(\lambda)$ (see [7, 10]). In practice we use the modified cross-periodogram statistic to avoid substantial bias near $\lambda = 0$. The new statistic is given by

$$\hat{I}_{NX}^{(R)}(\lambda) = \frac{1}{2\pi R} \hat{d}_N^{(R)}(\lambda)\, \overline{\hat{d}_X^{(R)}(\lambda)}, \qquad -\infty < \lambda < +\infty, \quad (9)$$

where

$$\hat{d}_N^{(R)}(\lambda) = d_N^{(R)}(\lambda) - \hat{p}_N \Delta^{(R)}(\lambda), \quad (10)$$

$$\hat{d}_X^{(R)}(\lambda) = d_X^{(R)}(\lambda) - \hat{\mu}_X \Delta^{(R)}(\lambda), \quad (11)$$

and

$$\Delta^{(R)}(\lambda) = \int_0^R \exp\{-i\lambda t\}\, dt. \quad (12)$$

We denote the estimates of the mean intensity of the point process and the mean value of the time series, respectively, by $\hat{p}_N$ and $\hat{\mu}_X$.

The definition for the cross-periodogram statistic was suggested by Jenkins [11] in the discussion of Bartlett's paper on the spectral analysis of point processes (see [12]). Higher order periodogram statistics can be defined in a similar way. For example, the third-order periodogram $I_{NXX}^{(R)}(\lambda\mu)$ is defined by

$$I_{NXX}^{(R)}(\lambda,\mu)=\frac{1}{(2\pi)^2R}d_N^{(R)}(\lambda)d_X^{(R)}(\lambda)\,\overline{d_X^{(R)}(\lambda+\mu)}\,, \qquad -\infty<\lambda,\mu<+\infty. \tag{13}$$

The method of estimating the cross-spectral density between $\{X(t)\}$ and $\{N(t)\}$ can be described as follows. We split the whole record of the data T into L disjoint subrecords each of length R so that $T=LR$. The cross-periodogram statistic is calculated in each subrecord, and the estimate of the cross-spectral density is then found by averaging the separate periodogram ordinates at each frequency. Mathematically this is expressed in the form

$$f_{NX}^{(LR)}(\lambda)=\frac{1}{L}\sum_{j=1}^{L}\hat{I}_{NX}^{(R)}(\lambda,j) \qquad \text{for } \lambda\neq 0, \tag{14}$$

where $\hat{I}_{NX}^{(R)}(\lambda,j)$ is the modified periodogram statistic between $\{X(t)\}$ and $\{N(t)\}$ for the jth sample. This estimate can be further improved by applying a weighting scheme as follows:

$$\hat{f}_{NX}^{(LR)}(\lambda_k)=\frac{1}{3}\left[f_{NX}^{(LR)}(\lambda_{k-1})+f_{NX}^{(LR)}(\lambda_k)+f_{NX}^{(LR)}(\lambda_{k-1})\right], \tag{15a}$$

$$\hat{f}_{NX}^{(LR)}(0)=\frac{1}{2}\left[f_{NX}^{(LR)}(0)+f_{NX}^{(LR)}(\lambda_1)\right], \tag{15b}$$

where

$$\lambda_k=\frac{2\pi k}{R} \qquad \text{and} \qquad k=1,2,\ldots,(R-1)/2.$$

ASSUMPTION 1

The stationary hybrid process $\{X(t),N(t)\}$, $-\infty<t<+\infty$, processes moments of all orders and satisfies the condition

$$\int\cdots\int(1+|u_l|)\left|c_{a_1,\ldots,a_j}(u_1,\ldots,u_{j-1})\right|du_1\cdots du_{j-1}<\infty,$$

where $c_{a_1,\ldots,a_j}(u_1,\ldots,u_{j-1})$ is the jth-order cumulant function of $\{X(t),N(t)\}$ and a_k, $k=1,2,\ldots,j$, being either $X(t)$ or $N(t)$; $l=1,2,\ldots,j-1$ and $j=2,3,\ldots$.

THEOREM 1

Let $\{X(t), N(t)\}$, $-\infty < t < +\infty$, be a stationary hybrid process satisfying Assumption 1. *Let $\hat{I}_{NX}^{(R)}(\lambda)$ be given by* (9). *Then*

$$\lim_{R\to\infty} E\{\hat{I}_{NX}^{(R)}(\lambda)\} = f_{NX}(\lambda) \qquad \text{for } \lambda \neq 0$$

and

$$\begin{aligned}\lim_{R\to\infty} \operatorname{cov}\{\hat{I}_{NX}^{(R)}(\lambda), \hat{I}_{NX}^{(R)}(\mu)\} &\\ = \delta\{\lambda-\mu\} f_{NN}(\lambda) f_{XX}(\lambda) &\\ + \delta\{\lambda+\mu\} f_{NX}(\lambda) f_{XN}(-\lambda) & \qquad \text{for } \lambda, \mu \neq 0.\end{aligned}$$

We suppose also that $2\lambda_j, \lambda_j + \lambda_k \neq 0$ for $1 \leqslant j < k \leqslant J$. Then $\hat{I}_{NX}^{(R)}(\lambda_j)$, $j = 1, \ldots, J$, are asymptotically independent $W_1^C(1, f_{NX}(\lambda_j))$.

$W_1^C(1, f_{NX}(\lambda_j))$ denotes the complex Wishart distribution with one degree of freedom and dimension 1 (see [13]). This distribution suggests that the modified cross-periodogram statistic is a poor estimate for the cross-spectral density (see Brillinger [7], p. 238). We proceed now to examine the asymptotic properties of $f_{NX}^{(LR)}(\lambda)$.

THEOREM 2

Let $\{X(t), N(t)\}$, $-\infty < t < +\infty$, be a stationary hybrid process satisfying Assumption 1. *Let $f_{NX}^{(LR)}(\lambda)$ be given by* (14). *Then*

$$\lim_{R\to\infty} E\{f_{NX}^{(LR)}(\lambda)\} = f_{NX}(\lambda) \qquad \text{for } \lambda \neq 0$$

and

$$\begin{aligned}\lim_{R\to\infty} \operatorname{cov}\{f_{NX}^{(LR)}(\lambda), f_{NX}^{(LR)}(\mu)\} &\\ = \delta\{\lambda-\mu\} \frac{f_{NN}(\lambda) f_{XX}(\lambda)}{L} &\\ + \delta\{\lambda+\mu\} \frac{f_{NX}(\lambda) f_{XN}(-\lambda)}{L} & \qquad \text{for } \lambda, \mu \neq 0.\end{aligned}$$

Furthermore, we suppose that $2\lambda_j, \lambda_j \pm \lambda_k \neq 0$ for $1 \leqslant j < k \leqslant J$. Then $f_{NX}^{(LR)}(\lambda_j)$, $j = 1, \ldots, J$, are asymptotically independent $L^{-1} W_1^C(L, f_{NX}(\lambda_j))$.

The proofs of the theorems are discussed in the Appendix.

Theorem 2 suggests that the distribution of $\hat{f}_{NX}^{(LR)}(\lambda)$ is asymptotically a multiple of a complex Wishart with $3L$ degrees of freedom and dimension 1. For large L the distribution of $\hat{f}_{NX}^{(LR)}(\lambda)$ will tend to a complex normal distribution.

The estimate $\hat{f}_{NX}^{(LR)}(\lambda)$ is a complex-valued function because the cross-covariance is an odd function and can be written in the form

$$\hat{f}_{NX}^{(LR)}(\lambda) = \operatorname{Re} \hat{f}_{NX}^{(LR)}(\lambda) + i \operatorname{Im} \hat{f}_{NX}^{(LR)}(\lambda). \tag{16}$$

A measure of the linear relationship between the time series $\{X(t)\}$ and the point process $\{N(t)\}$ is the coherence function, an estimate of which is given by

$$\begin{aligned} |\hat{R}_{NX}(\lambda)|^2 &= \frac{|\hat{f}_{NX}^{(LR)}(\lambda)|^2}{\hat{f}_{NN}^{(LR)}(\lambda)\hat{f}_{XX}^{(LR)}(\lambda)} \\ &= \frac{\left[\operatorname{Re} \hat{f}_{NX}^{(LR)}(\lambda)\right]^2 + \left[\operatorname{Im} \hat{f}_{NX}^{(LR)}(\lambda)\right]^2}{\hat{f}_{NN}^{(LR)}(\lambda)\hat{f}_{XX}^{(LR)}(\lambda)}, \end{aligned} \tag{17}$$

where $\hat{f}_{NN}^{(LR)}(\lambda)$ and $\hat{f}_{XX}^{(LR)}(\lambda)$ are the estimates of the power-spectral densities of $\{N(t)\}$ and $\{X(t)\}$, respectively. These estimates are obtained in the same way as the estimate of the cross-spectral density by splitting the whole record of the data into L disjoint subrecords. For more details refer to [7] and [9].

To test whether the coherence function is zero, we need to compute a 100α percent point of $|\hat{R}_{NX}^{(LR)}(\lambda)|^2$. This can be done by extending the formula of Abramovitz and Stegun [14, pp. 944–945] in the case of a hybrid process so that

$$\operatorname{Prob}\left\{|\hat{R}_{NX}(\lambda)|^2 < z\right\} = 1 - (1-z)^{s-1}, \qquad 0 < z < 1, \tag{18}$$

where $s = 3L$. It follows from (18) that

$$z = 1 - (1-\alpha)^{1/s-1}. \tag{19}$$

When $|\hat{R}_{NX}^{(LR)}(\lambda)|^2$ is less than z, we infer that the coherence function is zero, which implies that the point process $\{N(t)\}$ and the time series $\{X(t)\}$ are uncorrelated at all lags.

In the next section we concentrate on the estimate of the cross-covariance and examine its asymptotic distribution.

3. THE ESTIMATE OF THE CROSS-COVARIANCE FUNCTION AND ITS ASYMPTOTIC DISTRIBUTION

An estimate of the cross-covariance function can be obtained by taking the inverse Fourier transform of (6) and by inserting an estimate of the cross-spectral density in it. This is expressed as

$$\hat{c}_{NX}(u) = \frac{2\pi}{Q_R} \sum_j \hat{f}_{NX}^{(LR)}(\lambda_j) \exp\{i\lambda_j u\}, \quad j = \pm 1, \pm 2, \ldots, \pm \frac{Q_R - 1}{2}. \tag{20}$$

By using (16) in (20) we have

$$\hat{c}_{NX}(u) = \frac{4\pi}{Q_R} \sum_{j=1}^{(Q_R-1)/2} \Big[\operatorname{Re} \hat{f}_{NX}^{(LR)}(\lambda_j) \cos \lambda_j u - \operatorname{Im} \hat{f}_{NX}^{(LR)}(\lambda_j) \sin \lambda_j u\Big]. \tag{21}$$

This relation will be used in the computation of $\hat{c}_{NX}(u)$, as we shall see later.

We examine now the asymptotic distribution of the estimate of the cross-covariance function.

THEOREM 3

Let $\{X(t), N(t)\}$, $-\infty < t < +\infty$, be a stationary hybrid process satisfying Assumption 1. *Let $\hat{c}_{NX}(u)$ be given by* (20) *and $M = 3L$. Then $\hat{c}_{NX}(u)$ is asymptotically normal with first- and second-order moments given by*

$$E\{\hat{c}_{NX}(u)\} = c_{NX}(u) + O(Q_R^{-1} M^{-1} R^{-1}),$$

$$\begin{aligned}\operatorname{cov}\{\hat{c}_{NX}(u), \hat{c}_{NX}(v)\} &= \frac{2\pi}{MQ_R}\Big[\int f_{NN}(\lambda) f_{XX}(\lambda) \exp\{i\lambda(u-v)\}\, d\lambda \\ &\quad + \int f_{NX}(\lambda) f_{XN}(-\lambda) \exp\{i\lambda(u+v)\}\, d\lambda\Big] \\ &\quad + O(M^{-1}Q_R^{-2}) + O(M^{-1}R^{-1}) + O(M^{-1}Q_R^{-2}R^{-1}).\end{aligned}$$

Theorem 3 suggests the construction of an approximate 95% confidence interval for $\hat{c}_{NX}(u)$ under the hypothesis of independence between the time series $\{X(t)\}$ and the point process $\{N(t)\}$. Under the hypothesis of independence the distribution of $\hat{c}_{NX}(u)$ is asymptotically normal with mean 0 and variance given by

$$\operatorname{Var}\{\hat{c}_{NX}(u)\} \cong \frac{2\pi}{MQ_R} \int f_{NN}(\lambda) f_{XX}(\lambda)\, d\lambda,$$

and hence the confidence limits of the 95% interval are

$$\pm 1.96 \left[\frac{2\pi}{MQ_R} \int f_{NN}(\lambda) f_{XX}(\lambda)\, d\lambda \right]^{1/2}. \tag{22}$$

In practical problems these limits are calculated by substituting the power-spectral densities $f_{NN}(\lambda)$ and $f_{XX}(\lambda)$ with their estimates.

Before we go on to discuss some practical examples we present an estimate for the cross-cumulant density of a stationary bivariate point process. The definitions of the second and higher order cumulant densities are given in [10, 15].

An estimate of the cross-cumulant can be obtained in the same way as the estimate of the cross-covariance function, that is,

$$\hat{q}_{NM}(u) = \frac{2\pi}{Q_R} \sum_j \hat{f}_{NM}^{(LR)}(\lambda_j) \exp\{i\lambda_j u\}, \qquad j = \pm 1, \pm 2, \ldots, \pm \frac{Q_R - 1}{2}, \tag{23}$$

where $\hat{f}_{NM}^{(LR)}(\lambda_j)$ is the estimate of the cross-spectral density, which is obtained from (15) if we substitute the component X with M (see [15]). The asymptotic 95% confidence limits for $\hat{q}_{NM}(u)$ under the hypothesis of independence between N and M are given by

$$\pm 1.96 \left[\frac{2\pi}{MQ_R} \int f_{NN}(\lambda) f_{MM}(\lambda)\, d\lambda \right]^{1/2}.$$

4. EXAMPLES

In this section we analyze three data sets from the field of neurophysiology in order to extract useful information about the complex physiological system called the muscle spindle. This control system is a receptor that plays an important role in the initiation of movement and the maintenance of posture.

The muscle spindle is a transducer that responds to different stimuli. Muscle spindles are parts of the skeletal muscles, which are concerned with posture or movement. Most skeletal muscles contain a number of these receptors, which lie parallel to extrafusal fibers. They consist of a number of specialized fibers lying parallel to each other and partially contained within a fluid-filled capsule of connective tissue. The fibers within a muscle spindle, known as intrafusal fibers, are considerably shorter than the extrafusal muscle fibers.

The effects of imposed stimuli on the intrafusal muscle fibers are transmitted to the spinal cord by the axons of sensory nerves closely associated with the muscle spindle. The (fine) terminal branches of the sensory axons form spirals around the central region of the intrafusal muscle fibers.

Detailed discussions of the structure of the muscle spindle and its functional role can be found in Boyd [16] and Matthews [17]. In what follows we consider that the muscle spindle is an element of a muscle known as the parent muscle.

When a muscle is held at a fixed length, the sensory axons from the muscle spindle generate action potentials at a constant rate, which depends upon the muscle length [9, 18]. The action potential is a localized voltage change that occurs across the membrane surrounding the nerve cell and axon, with amplitude approximately 100 mV and duration 1 ms. An increase in muscle length will increase the rate of discharge of action potentials in the sensory endings. The deformation of the intrafusal muscle fibers caused by length changes imposed on the parent muscle distorts the fine terminals of the sensory axons, and thus the rate of discharge of these axons is modified. The discharge of the fine terminals of the sensory axons is also modified by action potentials carried by the axons of a group of cells called gamma motor neurons. The bodies of these cells lie inside the spinal cord, while their long axons innervate the intrafusal fibers of the muscle spindles.

As we have already mentioned, the muscle spindle is considered to be a hybrid system involving a time series and a point process. Length changes imposed on the parent muscle are considered a realization of a time series, while the discharge of the sensory axons from the muscle spindle (known as the Ia response), which consists of a sequence of nerve pulses, is considered a realization of a point process. The aim of this work is to examine the behavior of the muscle spindle when it is affected by length changes imposed on the parent muscle and at the same time (A) no other stimulus is present or (B) a gamma stimulation is present.

In the experiments the tenuissimus muscle in anesthetized cats was used, and the responses of single sensory axons in dorsal root filaments were recorded. Length changes, with a normal distribution of amplitudes, and a flat power spectrum from 0 to 120 Hz at rms amplitudes of 20–40 μm were applied to the tenuissimus muscle by a servo-controlled muscle stretcher. Gamma motor neuron axons isolated in ventral root filaments were stimulated by sequences of pulses at twice threshold. The distribution of intervals between pulses was approximately exponential. Fifteen-second sequences of the Ia response were recorded in the cases A and B defined above.

Figure 1 describes the Ia response of the muscle spindle when length changes are present. In Figure 1A the histogram of the intervals between pulses of the Ia response is shown. The value of h was taken to be 2 ms. The histogram gives an idea of the distribution of the intervals. Figure 1B presents the scatter diagram of adjacent intervals between pulses of the Ia response. The scatter diagram shows whether adjacent intervals between events are correlated. Figure 1C presents the logarithm to base 10 of the proportion of the intervals between pulses longer than t. This quantity is the logarithm of the empirical survivor function (see [19]), and if it is a straight line then it indicates that the intervals between pulses follow an exponential distribution. Figure 1D illustrates a sequence of nerve impulses of the Ia response in the presence of length changes.

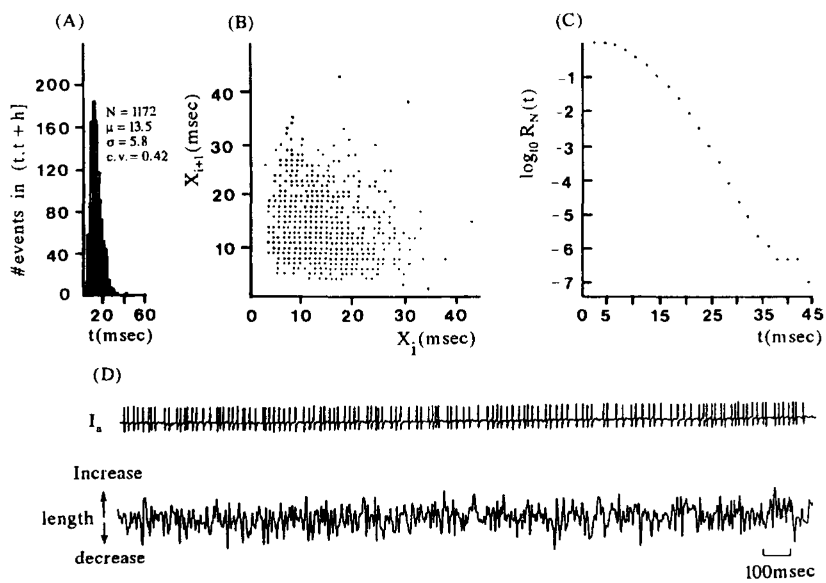


FIG. 1. Ia response when length changes are present. (A) Histogram of the intervals between pulses of the Ia response; (B) scatter diagram of adjacent intervals between pulses of the Ia response; (C) the logarithm of the proportion of the intervals between pulses longer than t; (D) sequence of nerve impulses of the Ia response.

Figure 2 describes the Ia response of the muscle spindle when length changes and a gamma stimulation (γ_s) are both present. The results in Figures 2A–C are similar to those discussed in Figures 1A–C. Figure 2D illustrates a sequence of nerve impulses of the Ia response in the presence of length changes and a gamma stimulation (γ_s).

Figure 3 describes the second-order properties of the length changes. Figure 3A shows the estimate of the autocovariance function obtained from the inverse Fourier transform of the estimated power spectral density. Figure 3B shows the estimate of the power spectral density obtained by splitting the whole record of the data $T = 15872$ ms into $L = 31$ disjoint subrecords, each of length $R = 512$ ms.

Figure 4 shows the estimates of the coherence function. The estimates of the cross-spectral density and the power-spectral densities needed for the estimation of the coherence function were obtained by

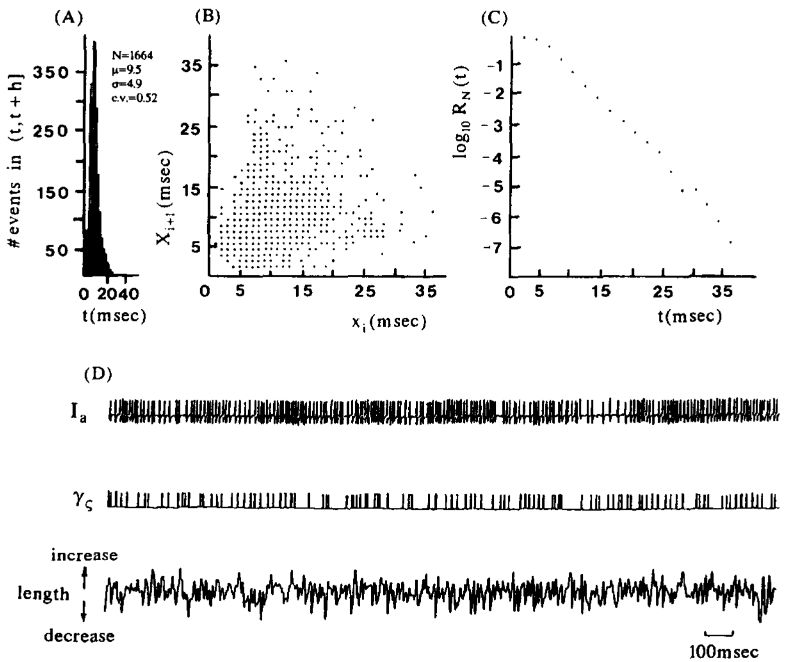


FIG. 2. Ia response when a gamma stimulation and length changes are present. (A) Histogram of the intervals between pulses of the Ia response; (B) scatter diagram of adjacent intervals between pulses of the Ia response; (C) the logarithm of the proportion of the intervals between pulses longer than t; (D) sequence of nerve impulses of the Ia response when a gamma stimulation (γ_s) and length changes are present.

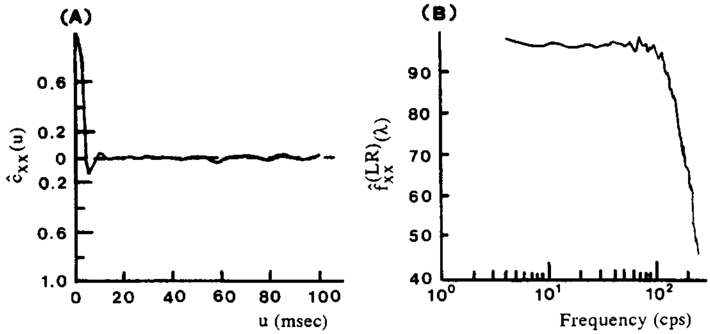


FIG. 3. Second-order properties of the length changes. (A) Estimate of the autocovariance function; (B) estimate of the power-spectral density.

splitting the whole record of the data into 31 disjoint subrecords each of length 512 ms. Figures 4A and B present the estimates of the coherence function when the muscle spindle is affected (A) by the presence of length changes only and (B) by the presence of length changes and a gamma motor neuron simultaneously. The dashed line in each figure corresponds to the 95% point of the coherence function obtained from Equation (19) when we set $s = 93$. The presence of the gamma stimulation reduces the value of the coherence function in the range 3–30 Hz, which implies that the length changes and the Ia response of the muscle spindle become uncorrelated. At higher frequencies the two figures remain almost the same.

Figure 5 presents the estimates of the cross-covariance function for the two cases A and B as described in Figure 4. These estimates are computed by using Equation (21) with $Q_R = 200$ and $M = 3L = 93$. The dashed line in the figures corresponds to zero, and the solid lines are the 95% confidence limits computed as

$$\pm \frac{1.96}{Q_R}\left[\frac{4\pi}{M}\sum_{j=1}^{(Q_R-1)/2} \hat{f}_{NN}^{(LR)}(\lambda_j)\hat{f}_{XX}^{(LR)}(\lambda_j)\right]^{1/2}.$$

The dashed line is the mean value of the estimate of the cross-covariance function, which, under the hypothesis of independence between N and X, is approximately zero. From Figures 5A and B we can see that the length changes and Ia response of the muscle spindle are

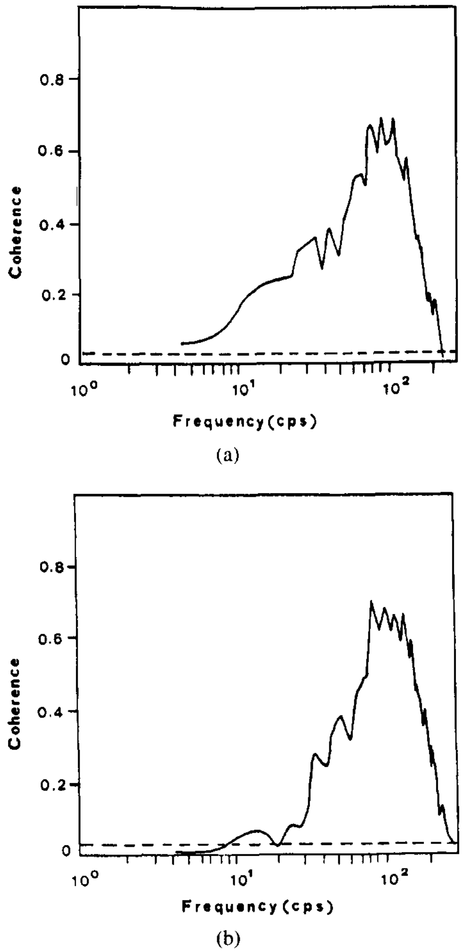
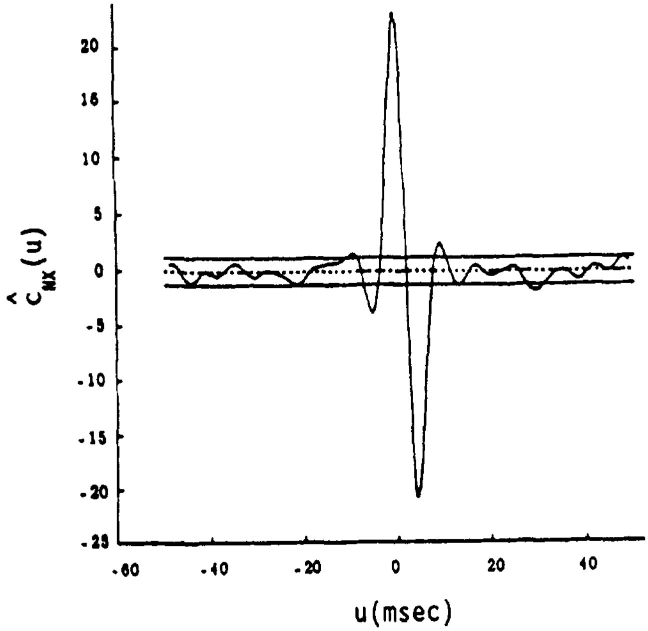


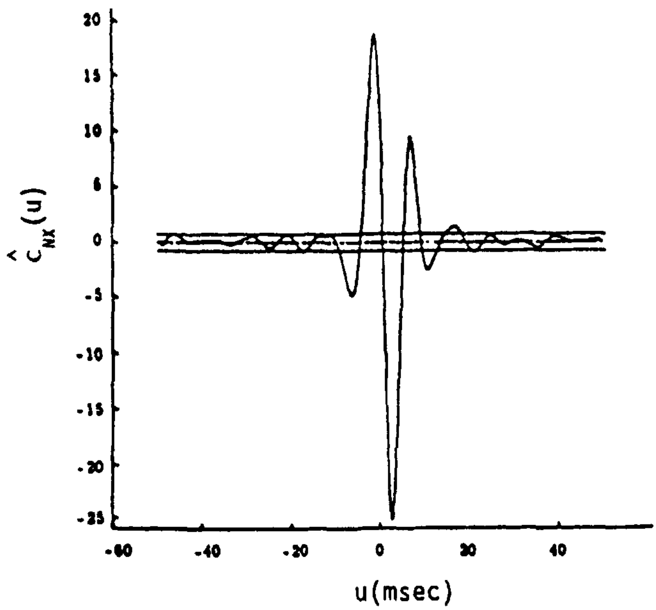
FIG. 4. Estimates of the coherence function (A) when length changes affect the muscle spindle; (B) when length changes and a gamma stimulation affect the muscle spindle simultaneously.

related for small lags in the range 0–10 ms. In the case where the gamma stimulation is present, the relation between the length changes and the Ia response seems to change somehow and to last for a few milliseconds longer. The positive values of the estimates indicate that the change in the length of the muscle, on average, corresponds to an increase, whereas the negative values correspond to a decrease.

Figure 6 presents the estimates of the cross-cumulant density obtained by using (23). Figure 6A shows the estimate of the cross-



(a)



(b)

FIG. 5. Estimates of the cross-covariance function (A) when length changes affect the muscle spindle; (B) when length changes and a gamma stimulation affect the muscle spindle simultaneously.

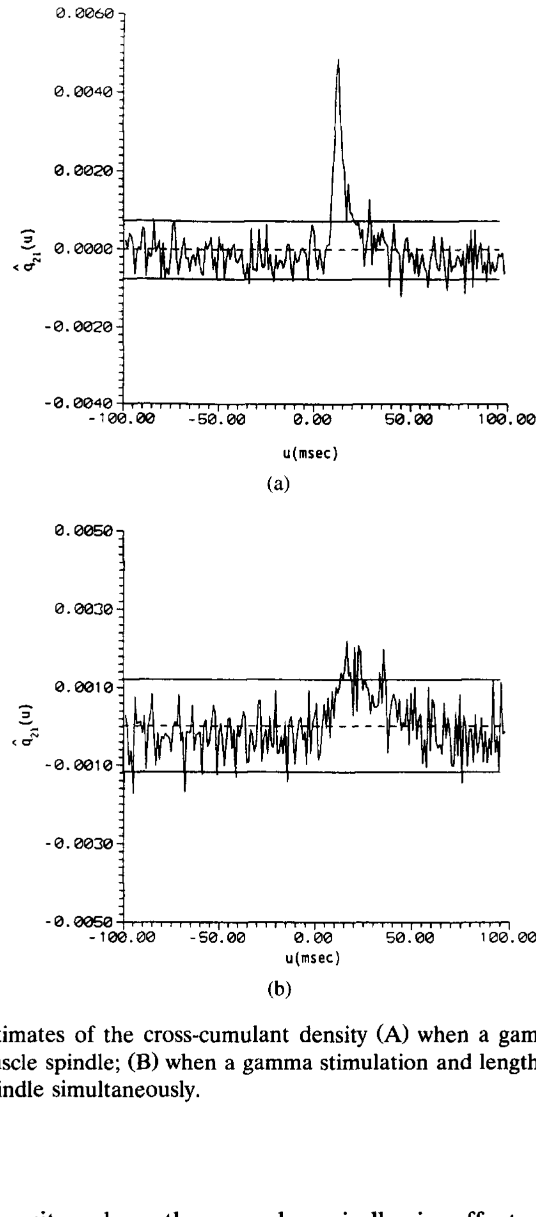


FIG. 6. Estimates of the cross-cumulant density (A) when a gamma stimulation affects the muscle spindle; (B) when a gamma stimulation and length changes affect the muscle spindle simultaneously.

cumulant density when the muscle spindle is affected only by a gamma stimulation, whereas Figure 6B shows the estimate of the cross-cumulant density when the muscle spindle is affected by a gamma stimulation and length changes simultaneously. The dashed line corresponds to zero, and the solid lines correspond to the 95% confidence limits computed by

$$\pm \frac{1.96}{Q_R}\left[\frac{4\pi}{M}\sum_{j=1}^{(Q_R-1)/2} \hat{f}_{NN}^{(LR)}(\lambda_j)\hat{f}_{MM}^{(LR)}(\lambda_j)\right]^{1/2}.$$

The dashed line is the mean value of the estimate of the cross-cumulant density, which, under the hypothesis of independence between N and M, is approximately zero. The relation between the gamma stimulation and the Ia response of the muscle spindle becomes weaker when the length changes are present.

By comparing Figures 5 and 6 we see that the length changes have an effect on the muscle spindle at small lags, whereas the gamma stimulation has an effect at large ones. This further suggests that the response of the muscle spindle to length changes takes place very quickly, whereas the response to the gamma stimulation is slower. A work is in preparation in which it will be shown that the information transmitted by the muscle spindle to the spinal cord can be separated into two parts, one at low frequencies (corresponding to gamma stimulation) and another at higher frequencies (corresponding to length changes) by applying a formal statistical test.

5. CONCLUSIONS

We have proposed an estimate for the cross-spectral density of a stationary hybrid process involving a time series and a point process in order to study a complex physiological system called the muscle spindle. The estimate of the cross-spectral density is obtained by splitting the whole record of the data into a number of disjoint subrecords. Estimates of the coherence function and the cross-covariance function can also be obtained by using the estimate of cross-spectral density. These estimates provide useful information for the properties of the muscle spindle in both the time and frequency domains. It is shown that the presence of a gamma stimulation reduces the effect of the length changes on the muscle spindle at low frequencies, whereas at higher frequencies it produces only a shift.

The methods presented here can be extended to estimate higher order spectral densities and higher order covariance densities, which will facilitate the study of the nonlinear behavior of the muscle spindle. This kind of work will be published in the near future.

I am most grateful to Professor G. P. Moore and Dr. J. R. Rosenberg for providing the data sets. I also thank Mr. D. S. Tsitsis for his invaluable help *in the preparation of this paper. Finally, I wish to express my gratitude to the two anonymous referees for their constructive and helpful comments, which led to the presentation of this paper.*

APPENDIX: PROOFS OF THEOREMS

PROOF OF THEOREM 1

It follows from the properties of complex random variables that

$$E\{\hat{d}_N^{(R)}(\lambda)\hat{d}_X^{(R)}(-\lambda)\}$$

$$= E\{d_N^{(R)}(\lambda), d_X^{(R)}(-\lambda)\} - p_N\mu_X|\Delta^{(R)}(\lambda)|^2 + O(R^{-1})$$

$$= \operatorname{cov}\{d_N^{(R)}(\lambda), d_X^{(R)}(\lambda)\} + O(R^{-1}) \qquad \text{for } \lambda \neq 0.$$

For the covariance of $d_N^{(R)}(\lambda)$ and $d_X^{(R)}(\lambda)$, we have

$$\operatorname{cov}\{d_N^{(R)}(\lambda), d_X^{(R)}(\lambda)\}$$

$$= \int_0^R \int_0^R \exp\{-i\lambda t\}\exp\{i\lambda s\}\operatorname{cov}\{dN(t), X(s)\}\, ds$$

$$= \int_0^R \int_0^R \exp\{-i\lambda(t-s)\}c_{NX}(t-s)\, dt\, ds$$

$$= \int_{-R}^R (R-|u|)c_{NX}(u)\exp\{-i\lambda u\}\, du,$$

and using (9) we get

$$E\{\hat{I}_{NX}^{(R)}(\lambda)\}$$

$$= (2\pi)^{-1}\int_{-R}^R c_{NX}(u)\exp\{-i\lambda u\}\, du + O(R^{-1}) \qquad \text{for } \lambda \neq 0.$$

Hence

$$\lim_{R\to\infty} E\{\hat{I}_{NX}^{(R)}(\lambda)\} = f_{NX}(\lambda) \qquad \text{for } \lambda \neq 0.$$

Applying now the properties of cumulants we find

$$\begin{aligned}
&\operatorname{cov}\{\hat{d}_N^{(R)}(\lambda)\hat{d}_X^{(R)}(-\lambda), \hat{d}_N^{(R)}(\mu)\hat{d}_X^{(R)}(-\mu)\} \\
&\quad = \operatorname{cum}\{\hat{d}_N^{(R)}(\lambda)\hat{d}_X^{(R)}(-\lambda), \hat{d}_N^{(R)}(-\mu)\hat{d}_X^{(R)}(\mu)\} \\
&\quad = \operatorname{cum}\{d_N^{(R)}(\lambda), d_X^{(R)}(-\lambda), d_N^{(R)}(-\mu), d_X^{(R)}(\mu)\} \\
&\qquad + \operatorname{cum}\{d_N^{(R)}(\lambda), d_N^{(R)}(-\mu)\}\operatorname{cum}\{d_X^{(R)}(-\lambda), d_X^{(R)}(\mu)\} \\
&\qquad + \operatorname{cum}\{d_N^{(R)}(\lambda), d_X^{(R)}(\mu)\}\operatorname{cum}\{d_X^{(R)}(-\lambda), d_N^{(R)}(-\mu)\} \\
&\qquad + \{\Delta^{(R)}(\lambda+\mu) + \Delta^{(R)}(\lambda-\mu) + \Delta^{(R)}(-\lambda+\mu) \\
&\qquad + \Delta^{(R)}(-\lambda-\mu) + 1\}O(1) \qquad \text{for } \lambda, \mu \neq 0 \\
&\quad = (2\pi)^2 R f_{NXNX}(\lambda, -\lambda, -\mu) + O(1) \\
&\qquad + (2\pi)^2 |\Delta^{(R)}(\lambda-\mu)|^2 f_{NN}(\lambda) f_{XX}(\lambda) + O(1) \\
&\qquad + (2\pi)^2 |\Delta^{(R)}(\lambda+\mu)|^2 f_{NX}(\lambda) f_{XN}(-\lambda) + O(1) \\
&\qquad + \{\Delta^{(R)}(\lambda+\mu) + \Delta^{(R)}(\lambda-\mu) + \Delta^{(R)}(-\lambda+\mu) \\
&\qquad + \Delta^{(R)}(-\lambda-\mu) + 1\}O(1) \qquad \text{for } \lambda, \mu \neq 0.
\end{aligned}$$

This leads to

$$\begin{aligned}
&\lim_{R\to\infty} \operatorname{cov}\{\hat{I}_{NX}^{(R)}(\lambda), \hat{I}_{NX}^{(R)}(\mu)\} \\
&\qquad = \delta\{\lambda-\mu\} f_{NN}(\lambda) f_{XX}(\lambda) + \delta\{\lambda+\mu\} f_{NX}(\lambda) f_{NX}(-\lambda), \\
&\hspace{28em} \lambda, \mu \neq 0,
\end{aligned}$$

since $|\Delta^{(R)}(\lambda)|^2/R^2 = \delta\{\lambda\}$ as $R\to\infty$, where $\delta\{\lambda\}$ is the Kronecker delta. It can be proved [6] that $\{\hat{d}_N^{(R)}(\lambda_j), \hat{d}_X^{(R)}(\lambda_j)\}$, $j = 1,\ldots,J$, are asymptotically independent $N_1^C(0, 2\pi R f_{NX}(\lambda_j))$ variates. Hence, by using a lemma of Brillinger [7], we prove that $\hat{I}_{NX}^{(R)}(\lambda_j)$, $j = 1,\ldots,J$, are asymptotically independent $W_1^C(1, f_{NX}(\lambda_j))$ variates.

PROOF OF THEOREM 2

It follows from Theorem 1 that

$$E\{\hat{I}_{NX}^{(R)}(\lambda, j)\} = f_{NX}(\lambda) + O(R^{-1}) \qquad \text{for } \lambda \neq 0;\ j = 1,\ldots,L.$$

Hence

$$\lim_{R\to\infty} E\{f_{NX}^{(LR)}(\lambda)\} = f_{NX}(\lambda).$$

For the covariance of $f_{NX}^{(LR)}(\lambda)$ and $f_{NX}^{(LR)}(\mu)$, we have

$$\begin{aligned}\operatorname{cov}\{f_{NX}^{(LR)}(\lambda), f_{NX}^{(LR)}(\mu)\} &= \operatorname{cum}\{f_{NX}^{(LR)}(\lambda), f_{NX}^{(LR)}(-\mu)\}\\ &= \frac{1}{L^2}\sum_{j=1}^{L}\sum_{k=1}^{L}\operatorname{cum}\{\hat{I}_{NX}^{(R)}(\lambda,j), \hat{I}_{NX}^{(R)}(-\mu,k)\}\\ &= \frac{1}{L^2}\sum_{j=1}^{L}\operatorname{cum}\{\hat{I}_{NX}^{(R)}(\lambda,j), \hat{I}_{NX}^{(R)}(-\mu,j)\}\end{aligned}$$

since the jth and kth subrecords are disjoint. The last relation leads to the required result, because

$$\begin{aligned}&\operatorname{cum}\{\hat{I}_{NX}^{(R)}(\lambda,j), \hat{I}_{NX}^{(R)}(-\mu,j)\}\\ &\quad = \delta\{\lambda-\mu\} f_{NN}(\lambda) f_{XX}(\lambda) + \delta\{\lambda+\mu\} f_{NX}(\lambda) f_{NX}(-\lambda) + O(R^{-1}).\end{aligned}$$

The final result follows from the fact that the estimate $f_{NX}^{(LR)}(\lambda)$ is the average of the asymptotically independent variates $\hat{I}_{NX}(\lambda_j)$, $j = 1,\ldots,L$.

PROOF OF THEOREM 3

It follows from Theorem 2 that

$$E\{\hat{c}_{NX}(u)\} = c_{NX}(u) + O(M^{-1}R^{-1}Q_R^{-1})$$

and

$$\begin{aligned}&\operatorname{cov}\{\hat{c}_{NX}(u), \hat{c}_{NX}(v)\}\\ &\quad = \frac{2\pi}{MQ_R}\left[\int f_{NN}(\lambda) f_{XX}(\lambda)\exp\{i\lambda(u-v)\}\,d\lambda\right.\\ &\qquad\left. + \int f_{NX}(\lambda) f_{XN}(-\lambda)\exp\{i\lambda(u+v)\}\,d\lambda\right]\\ &\qquad + O(M^{-1}R^{-1}) + O(M^{-1}Q_R^{-2}) + O(M^{-1}R^{-1}Q_R^{-2}).\end{aligned}$$

The asymptotic result for the covariance holds if $Q_R \to \infty$, $Q_R R^{-1} \to 0$ as $R \to \infty$. The asymptotic normality follows from the fact that

$$\begin{aligned}&\operatorname{cum}\{\hat{c}_{NX}(u_1),\ldots,\hat{c}_{NX}(u_J)\}\\&\quad=\left(\frac{2\pi}{Q_R}\right)^J \sum\cdots\sum \operatorname{cum}\{\hat{f}_{NX}^{(LR)}(\lambda_1),\ldots,\hat{f}_{NX}^{(LR)}(\lambda_J)\}\\&\qquad\times\exp\{i(\lambda_1 u_1+\cdots+\lambda_J u_J)\}.\end{aligned}$$

Moreover,

$$\begin{aligned}&\operatorname{cum}\{\hat{f}_{NX}^{(LR)}(\lambda_1),\ldots,\hat{f}_{NX}^{(LR)}(\lambda_J)\}\\&\quad=\frac{1}{M^J}\sum_j \operatorname{cum}\{\hat{I}_{NX}^{(R)}(\lambda_1,j),\ldots,\hat{I}_{NX}^{(R)}(\lambda_J,j)\}.\end{aligned}$$

From these relations we have

$$\operatorname{cum}\{\hat{c}_{NX}(u_1),\ldots,\hat{c}_{NX}(u_J)\}=O(M^{-J+1}Q_R^{-J+1}),$$

since

$$\begin{aligned}&\operatorname{cum}\{\hat{I}_{NX}^{(R)}(\lambda_1,j),\hat{I}_{NX}^{(R)}(\lambda_2,j),\ldots,\hat{I}_{NX}^{(R)}(\lambda_J,j)\}\\&\quad=O(1)\qquad\text{for } \lambda_j\neq 0;\ j=1,2,\ldots,J.\end{aligned}$$

The last relation can be proved similarly to the case of ordinary time series [7, p. 418]. Hence

$$Q_R^{J/2}\operatorname{cum}\{\hat{c}_{NX}(u_1),\hat{c}_{NX}(u_2),\ldots,\hat{c}_{NX}(u_J)\}\to 0 \qquad \text{as } R\to\infty \text{ and } J>2.$$

This gives the required result because the cumulants of order higher than 2 are asymptotically zero.

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