Simplified option pricing techniques

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Abstract

Most of the currently known option pricing techniques utilize the underlying asset price and strike price, its volatility and time to maturity, as well as the risk free-rate. However, both the volatility and the risk-free rate are anticipated via the price move of the underlying asset. Looking at the same time at the Brownian motion, on which we base the modeling of the underlying asset price-move so as to value an option, we realize that its volatility is captured by the time to maturity. Moreover, the value of an option increases both as the volatility and time to maturity increase.

These observations make us believe that we could find simplified option pricing formulas depending on the underlying asset (price and strike price) and the time to maturity only. The advantage of the approach is that less simplifying assumptions are needed and much simpler methods are produced.

In this paper we provide alternative formulas for pricing European and American type options. We test our formulas against the Greek stock and derivatives market by applying the appropriate hypothesis testing.

Keywords: option pricing, time to maturity, call and put option, volatility, hypothesis testing

1 Introduction

The most well known option pricing approach for a European call or put option in continuous time is the Black-Scholes formula. This states that for a European call option *c*, maturing at time *T*, with strike price *k*, written on a non-dividend paying stock *S*, with volatility *σ*, when the risk-free rate is *r*, the value of the option is given by:

$$c_t = S \cdot N(d_1) - k \cdot e^{-r(T-t)} \cdot N(d_2),$$

where *N* denotes the standard normal distribution probability function and

$$d_1 = \frac{\ln(S/k) + (r + \sigma^2/2) \cdot (T-t)}{\sigma \cdot \sqrt{T-t}},\ d_2 = d_1 - \sigma \cdot \sqrt{T-t}.$$

The value of a put option can be derived via the put-call parity:

$$p_t + S = c_t + k \cdot e^{-r \cdot (T-t)}.$$

The value of an American call is identical to the value of a European call, provided no dividend is paid during the life of the option, as early exercise in never preferable to

retaining the option until maturity. However, with this approach not much can be said about the value of an American put option. Black-Scholes formula cannot be used and one has to rely on other methods such as binomial trees.
In this paper we provide alternative methods for pricing European and American call and put options. Our contribution lies in the simplification attempted in the models developed. Such simplification is feasible due to our observation that the value of the option can be derived as a function of the underlying stock price, the strike price and time to maturity. This route is supported by the fact that both the risk-free rate and the volatility of the stock are captured by the move of the underlying stock price. Moreover, looking at the properties of the Brownian motion, widely used to map the move of the stock price, we realize that volatility is well depicted by time. Last but not least, the value of an option is an increasing function both of time and volatility.

As a matter of fact, without doubting the success and beauty of the Black-Scholes pricing formula, one should not ignore that it uses some quite strong assumptions; if lifted it is not secured that the output will still hold true.

Following the above rationale we feel that by properly inserting time we can derive "nice and easy" option pricing techniques.

Literature review needs to be inserted to explain that the approach recommended is a prototype…

2 Models and formulas

We first focus at *European call options*. Would we find the value of a European call option with any model, the value of the respective European put option is given via the put-call parity. Hence, we will not make any specific mention to the latter.

We consider several alternative models. We will test their output vs. observed option prices. All our models use as input the particulars of the option with the maximum time to maturity, denoted by *Tmax*.

- We assume that the market price of the option that matures in *Tmax* is known and even more we accept the market prices it properly for each time instance $0 \leq t \leq T$. This is denoted by $c_t^{\max}$.
- Depending on the formula we investigate, we could restrict the assumption to $c_0^{\max}$, i.e. the price of the option for *t=0*.

Looking at the boundary conditions at *t=0* and *t=T*, we conclude that the alternative formulas considered should incorporate the value of the option with the maximum time to maturity, as well as the payoff of the option under investigation at maturity. Namely, the value of any European call option at maturity, denoted by c_T^E, satisfies the condition

$$c_T^E = \max(S_T - k; 0).$$

On top, we recall that when a call option is written usually the intrinsic value is zero (*0*), as the stock price is less than the strike price ($S_0 \le k$). As a consequence, it is the time value that matters.

We also, do not forget that the price of an option is nonnegative and should not exceed the price of the underlying stock, i.e.

$$0 \le c_t^E \le S_t$$

2.1 Formula introducing the time and the stock price

This is the simplest first cut and it is given by

$$c_t^E = \min\{[\max(S_t - k;0) + \frac{T-t}{T\max} \cdot S_t]; S_t\}$$

One can quickly verify that for *t=0* - and assuming that $S_0 \le k$ - the value of the option is nothing but the time value

$$c_0^E = \frac{T}{T\max} \cdot S_t\,.$$

At *t=T*, we receive what we expected, namely

$$c_T^E = \max(S_T - k;0)\,.$$

This formula can be refined though so that it properly values also the option with the longest time to maturity.

2.2 Formula introducing the time and the longest maturity option

With the notation used in the first paragraph of the models and formulas section we insert in the pricing formula the price of the longest-maturity option, replacing the stock price and accepting that the market prices it correctly. This gives

$$c_t^E = \min\{[\max(S_t - k;0) + \frac{T-t}{T\max} \cdot c_t^{\max}]; S_t\}\,,$$

where $c_t^{\max}$ is the value of the option maturing at *Tmax* at time *t*.

We can easily see that for *t=0* - and assuming once and again that $S_0 \le k$ - that the value of the option becomes

$$c_0^E = \frac{T}{T\max} \cdot c_0^{\max}\,.$$

At *t=T*, again

$$c_T^E = \max(S_T - k;0)\,.$$

The improvement is that it works though for the longest maturing option as well, for which *T=Tmax*. Observe that for *t=0*, we get

$$c_0^E = \frac{T\max}{T\max} \cdot c_0^{\max} = c_0^{\max}\,,$$

as anticipated. At the same time for *t=T=Tmax,* we receive

$$c_{T\max}^E = \max(S_{T\max} - k;0)\,,$$

again as expected for the payoff of a European call option at maturity.

2.3 Formula introducing the square root of time and the longest maturity option

Looking at the generalized Brownian motion we see that the square root of time is present next to the stochastic term that maps the volatility. This makes as consider the square root of time instead of time itself. Our formula thus becomes

$$c_t^E = \min\{[\max(S_t - k;0) + \sqrt{\frac{T-t}{T\max}} \cdot c_t^{\max}];S_t\}$$

We readily see that

$$c_0^E = \sqrt{\frac{T}{T\max}} \cdot c_0^{\max}\text{, for } t=0$$

$$c_T^E = \max(S_T - k;0)\text{, for } t=T.$$

On top, when *T=Tmax*

$$c_0^E = \sqrt{\frac{T\max}{T\max}} \cdot c_0^{\max} = c_0^{\max}\text{, for } t=0$$

$$c_{T\max}^E = \max(S_{T\max} - k;0)\text{, for } t=T=T\max.$$

2.4 Formula introducing the volatility and the risk-free rate

Although our approach primarily lies in the use of time, would we want to consider at the same time the volatility and the risk-free rate, that would be in the following format

$$c_t^E = \min\{[\max(S_t - k;0) + \sqrt{\frac{T-t}{T\max}} \cdot c_t^{\max} \cdot (1 + (r + \frac{\sigma^2}{2}) \cdot \sqrt{T\max - T})];S_t\}.$$

As before we see that

$$c_0^E = \sqrt{\frac{T}{T\max}} \cdot c_0^{\max} \cdot (1 + (r + \frac{\sigma^2}{2}) \cdot \sqrt{T\max - T})\text{, for } t=0$$

$$c_T^E = \max(S_T - k;0)\text{, for } t=T.$$

On top, when *T=Tmax*

$$c_0^E = \sqrt{\frac{T\max}{T\max}} \cdot c_0^{\max} \cdot (1 + (r + \frac{\sigma^2}{2}) \cdot \sqrt{T\max - T\max}) = c_0^{\max}\text{, for } t=0$$

$$c_{T\max}^E = \max(S_{T\max} - k;0)\text{, for } t=T=T\max.$$

2.5 Formula introducing the square root of time and the longest maturity option at t=0

Would we want to assume that the market prices properly the longest maturity option at time t=0, i.e. when the option of interest is written, then we modify our previous formula as follows

$$c_t^E = \min\{[\max(S_t - k;0) + \sqrt{\frac{T-t}{T\max}} \cdot c_0^{\max}];S_t\}.$$

We readily see that

$$c_0^E = \sqrt{\frac{T}{T\max}} \cdot c_0^{\max}\text{, for } t=0$$

$$c_T^E = \max(S_T - k;0)\text{, for } t=T.$$

On top, when *T=Tmax*

$$c_0^E = \sqrt{\frac{T\max}{T\max}} \cdot c_0^{\max} = c_0^{\max}\text{, for } t=0$$

$$c_{T\max}^E = \max(S_{T\max} - k;0)\text{, for } t=T=Tmax.$$

2.6 Formula introducing the volatility and the risk-free rate – take II

As in section 2.4 a variation that introduces the volatility and the risk-free rate is given by

$$c_t^E = \min\{[\max(S_t - k;0) + \sqrt{\frac{T-t}{T\max}} \cdot c_0^{\max} \cdot (1 + (r + \frac{\sigma^2}{2}) \cdot \sqrt{T\max - T})];S_t\}$$

As before we observe that

$$c_0^E = \sqrt{\frac{T}{T\max}} \cdot c_0^{\max} \cdot (1 + (r + \frac{\sigma^2}{2}) \cdot \sqrt{T\max - T})\text{, for } t=0$$

$$c_T^E = \max(S_T - k;0)\text{, for } t=T.$$

On top, when *T=Tmax*

$$c_0^E = \sqrt{\frac{T\max}{T\max}} \cdot c_0^{\max} \cdot (1 + (r + \frac{\sigma^2}{2}) \cdot \sqrt{T\max - T\max}) = c_0^{\max}\text{, for } t=0$$

$$c_{T\max}^E = \max(S_{T\max} - k;0)\text{, for } t=T=Tmax.$$

All the aforementioned formulas work when $S_0 \leq k$. What if this is not the case though? In the next tranche of formulas we attempt to tackle this.

2.7 Formula introducing the square root of time and the longest maturity option at t=0 for $S_0 = k$

Let $\tilde{c}_0^{\max}$ be the value of the option at *t=0* for which $S_0=k$ and *T=Tmax*. We let this price be our input and set:

$$c_t^E = \min\{[\max(S_t - k;0) + \sqrt{\frac{T-t}{T\max}} \cdot \tilde{c}_0^{\max}];S_t\}$$

All previous formulas having $c_0^{\max}$ could be altered to include $\tilde{c}_0^{\max}$ instead.

What does this give us for our usual boundary conditions? We easily see that

$$c_0^E = \min\{[\max(S_0 - k;0) + \sqrt{\frac{T}{T\max}} \cdot \tilde{c}_0^{\max}];S_0\}\text{, for } t=0$$

$$c_T^E = \max(S_T - k;0)\text{, for } t=T.$$

On top, when *T=Tmax*

$$c_0^E = \min\{[\max(S_0 - k;0) + \tilde{c}_0^{\max}];S_0\}\text{, for } t=0$$

$$c_{T\max}^E = \max(S_{T\max} - k;0)\text{, for } t=T=Tmax.$$

If $S_0=k$ and *T=Tmax,* then the former becomes

$$c_0^E = \tilde{c}_0^{\max},$$

which is probably expected.

2.8 Formula introducing the square root of time and the longest maturity option when S_t=k

Let $\tilde{c}_t$ denote the options for which S_t=*k*. Let $\tilde{c}_t^{\max}$ denote the option for which S_t=*k* and *T*=*Tmax*. We then use it as input and set:

$$c_t^E = \min\{[\max(S_t - k;0) + \sqrt{\frac{T-t}{T\max - t}} \cdot \tilde{c}_t^{\max}];S_t\}$$

All previous formulas having $c_t^{\max}$ could be altered to include $\tilde{c}_t^{\max}$ instead.

The boundary conditions become

$$c_0^E = \min\{[\max(S_0 - k;0) + \sqrt{\frac{T}{T\max}} \cdot \tilde{c}_0^{\max}];S_0\}, \text{ for } t=0$$

$$c_T^E = \max(S_T - k;0), \text{ for } t=T.$$

When *T*=*Tmax*

$$c_0^E = \min\{[\max(S_0 - k;0) + \tilde{c}_0^{\max}];S_0\}, \text{ for } t=0$$

$$c_{T\max}^E = \max(S_{T\max} - k;0), \text{ for } t=T=T\max.$$

If S_t=*k* and *T*=*Tmax,* then the formula yields

$$c_t^E = \tilde{c}_t^{\max},$$

showing that the formula prices properly the option used as input.

2.9 Formula introducing time, the square root of time and the longest maturity option at t=0

A global way to overcome our problematic is given by introducing the time elapsed in the pricing model. Such a route is indicated again by the Ito process followed by the stock price where time elapsed is part of the drift term. Our model yields then a price of

$$c_t^E = \min\{[\max(S_t - k;0)\frac{t}{T} + \sqrt{\frac{T-t}{T\max}} \cdot c_0^{\max}];S_t\}.$$

This formula overcomes the problematic described above, as there is no need to assume that $S_0 \le k$. The boundary conditions become

$$c_0^E = \sqrt{\frac{T}{T\max}} \cdot c_0^{\max}, \text{ for } t=0$$

$$c_T^E = \max(S_T - k;0), \text{ for } t=T.$$

On top, when *T*=*Tmax*

$$c_0^E = \sqrt{\frac{T\max}{T\max}} \cdot c_0^{\max} = c_0^{\max}, \text{ for } t=0$$

$$c_{T\max}^E = \max(S_{T\max} - k;0), \text{ for } t=T=T\max.$$

Variants of the above formula, matching what is indicated by other alternatives described above can be constructed.

What about *American call options*? We again try a series of formulas. The type considered is as follows

$$c_t^A = c_t^E + \frac{T}{T\max}(c_{t,\max}^A - c_t^E)$$

where c_t^A denotes the price of the American call option at time t and $c_{t,\max}^A$ stands for the historical maximum price of the American call option.

Observe that for $t=T$, would the call be on a non-dividend paying stock, we know that early exercise is not optimal. Hence at maturity

$$c_{T,\max}^A = c_T^E,$$

giving that

$$c_T^A = c_T^E.$$

In case of a put option, where early exercise could be optimal, $p_{t,\max}^A$ denotes the historical maximum of the American put at time t and $p_{T,\max}^A$ corresponds to the value at the time of the exercise. Thus,

$$\begin{aligned} p_t^A &= p_t^E + \frac{T}{T\max}\cdot(p_{t,\max}^A - p_t^E) \\ &= p_t^E \cdot (1 - \frac{T}{T\max}) + \frac{T}{T\max}\cdot p_{t,\max}^A \\ &\geq p_t^E \cdot (1 - \frac{T}{T\max}) + \frac{T}{T\max}\cdot p_t^E = p_t^E \end{aligned}$$

being what was expected.

As a matter of fact, when $T=Tmax$

$$p_t^A = p_{t,\max}^A.$$

This needs to be further looked at.

3 Results

We now test our models versus data from the Athens Stock Exchange. For this we consider call options on the FTSE/ Large Cap index.

Preliminary examples using the Black-Scholes formula (instead of actual data) show that our models produce “good results” - in terms of proximity – when the option is at the money. The best performance is exhibited by the formulas of sections 2.5, 2.6 and 2.9.

This needs however to be further elaborated and actually tested by employing the appropriate tests[1].

[1] *This section needs significant work as soon as complete data is available.*

4 Further research

In this article we worked primarily on the pricing of call and put options, European and American type on a non-dividend paying underlying asset. Further research could focus on

- The pricing of other types of options/ derivatives, even when analytical solutions are not in place.
- The pricing of options on dividend-paying underlying assets.
- The calculation of implied volatility and its comparison with the observed one.
- Potentially pricing insurance policies, realizing that they basically function "as an option". What needs to be embedded here is the probability of the occurrence of the insured risk.

5 Conclusions

In our study we managed to develop models that can be used to price options in a simple way, capitalizing on the properties of time as a proxy measure of volatility. We presented a series of formulas and we ranked them based on the proximity of their output to the actual option premium. As a matter of fact the best performing ones are those of sections 2.5, 2.6 and 2.9.

We trust that such an approach opens a route that can be further exploited and gives room for experimenting with other models as well.

References

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